greater than 4. Thus the positive difference between <u>any</u> pair of perfect squares is 3 or greater than 4. Therefore $c^2 - a^2$ cannot equal 4 if $c, a \in \mathbf{N}$. Therefore $a^2 + 4$ cannot be a perfect square and $\frac{a+\sqrt{a^2+4}}{2}$ and $\frac{a-\sqrt{a^2+4}}{2}$ are irrational whenever a is a positive integer.

We denote these 2 irrational conjugates by

$$v_a = \frac{a + \sqrt{a^2 + 4}}{2} \text{ and } \overline{v_a} = \frac{a - \sqrt{a^2 + 4}}{2}$$

noting that $v_1 = \varphi$, the golden ratio $\frac{1+\sqrt{5}}{2} \approx 1.618034$, and $\overline{v_1} = \frac{1-\sqrt{5}}{2} \approx -0.618034$. One might refer to the more general v_a as the "Golden v_a – ratio".

The following Theorem 1 will imply and generalize Equations (1) and (3):

(1) $\varphi^n = F_n\varphi + F_{n-1}$

(3) $x^n - F_n x - F_{n-1} = (x^2 - x - 1)(x^{n-2} + x^{n-3} + \cdots + F_{n-2}x + F_{n-1})$

Theorem 1: For any positive integer a,

(a) The equation with integer coefficients, $x^n = xF_n^a + F_{n-1}^a$ has solutions $v_a = \frac{a+\sqrt{a^2+4}}{2}$ and $\overline{v_a} = \frac{a-\sqrt{a^2+4}}{2}$ for all positive integers $n \geq 2$.

(b) The polynomial $x^n - xF_n^a - F_{n-1}^a$ factors as $x^n - xF_n^a - F_{n-1}^a = (x^2 - ax - 1)(x^{n-2} + ax^{n-3} + \cdots + xF_{n-2}^a + F_{n-1}^a)$ in which the latter factor has coefficients that are consecutive (increasing) elements of the sequence $\mathbf{F}_a$.

<u>Examples</u>: For $a = 2$:
The sequence (as noted previously) begins $1, 2, 5, 12, 29, 70, \ldots$ while
$v_2 = \frac{2+\sqrt{8}}{2} = 1 + \sqrt{2}$ and $\left(1 + \sqrt{2}\right)^6 = \left(1 + \sqrt{2}\right)70 + 29$
since both equal $99 + 70\sqrt{2}$.
Also $\overline{v_2} = 1 - \sqrt{2}$ and $\left(1 - \sqrt{2}\right)^6 = \left(1 - \sqrt{2}\right)70 + 29$
since both equal $99 - 70\sqrt{2}$.
Additionally, $x^6 - 70x - 29 = (x^2 - 2x - 1)(x^4 + 2x^3 + 5x^2 + 12x + 29)$.

For $a = 3$: The sequence begins $1, 3, 10, 33, 109, \ldots$ while
$v_3 = \frac{3+\sqrt{13}}{2}$, and $\left(\frac{3+\sqrt{13}}{2}\right)^5 = 109\left(\frac{3+\sqrt{13}}{2}\right) + 33$.
Also $\overline{v_3} = \frac{3-\sqrt{13}}{2}$ and $\left(\frac{3-\sqrt{13}}{2}\right)^5 = 109\left(\frac{3-\sqrt{13}}{2}\right) + 33$.
In addition, $x^5 - 109x - 33 = (x^2 - 3x - 1)(x^3 + 3x^2 + 10x + 33)$.

For $a = 6$: The sequence begins $1, 6, 37, 228, 1405, \ldots$ while
$v_6 = \frac{6+\sqrt{40}}{2} = 3 + \sqrt{10}$, and $\left(3 + \sqrt{10}\right)^4 = 228\left(3 + \sqrt{10}\right) + 37$.
Also $\overline{v_6} = 3 - \sqrt{10}$ and $\left(3 - \sqrt{10}\right)^4 = 228\left(3 - \sqrt{10}\right) + 37$.
Additionally, $x^4 - 228x - 37 = (x^2 - 6x - 1)(x^2 + 6x + 37)$.

We will give the polynomials in the preceding examples having (consecutive) terms of the Fibonacc–a Sequence as coefficients the name: *Fibonacc–a Polynomials*.

Definition: Polynomials in one variable for which the coefficients are consecutive integers in the Fibonacc$-a$ sequence are termed Fibonacc$-a$ Polynomials.

Proof of Theorem 1: (a) Proceeding by natural induction, we have by the discussion preceding the statement of this theorem that $x^2 - ax - 1 = 0$ if and only if $x = \frac{a+\sqrt{a^2+4}}{2}$ or $x = \frac{a-\sqrt{a^2+4}}{2}$, which can be verified using the quadratic formula. Since $F_2^a = a$ and $F_1^a = 1$, it follows for $n = 2$ that the equation $x^2 - ax - 1 = 0$ is equivalent to $x^n = xF_n^a + F_{n-1}^a$. So Part (a) is true when $n = 2$. If this equation is not satisfied by v_a and $\overline{v_a}$ for all $n \geq 2$, then there must be a smallest positive integer $m > 2$ for which the equation $r^m = rF_m^a + F_{m-1}^a$ is not true for $r = \frac{a+\sqrt{a^2+4}}{2}$ or $r = \frac{a-\sqrt{a^2+4}}{2}$. Therefore the equation is true when $n = m - 1 \geq 2$ for both $r = v_a$ and $\overline{v_a}$. Therefore

$$r^{m-1} = rF_{m-1}^a + F_{m-2}^a \text{ .}$$

Multiplying both sides by r yields the equation

$$r^m = r^2 F_{m-1}^a + rF_{m-2}^a \text{ .}$$

Substituting $ar + 1$ for r^2 (since $r^2 = ar + 1$ from the base case) yields

$$r^m = (ar + 1)F_{m-1}^a + rF_{m-2}^a \text{ .}$$

Collecting terms with respect to r gives that

$$r^m = r(aF_{m-1}^a + F_{m-2}^a) + F_{m-1}^a$$

The recursive definition for F_m^a is

$$F_m^a = aF_{m-1}^a + F_{m-2}^a$$

which may be substituted into the previous equation to give

$$r^m = rF_m^a + F_{m-1}^a \text{ .}$$

But this contradicts a logical consequence of the assumption that $x^n = xF_n^a + F_{n-1}^a$ is not true for all $n \geq 2$ whenever $x = \frac{a+\sqrt{a^2+4}}{2}$ or $x = \frac{a-\sqrt{a^2+4}}{2}$.

http://dx.doi.org/10.29322/IJSRP.13.02.2023.p13450

Thus we have

$$r^n = rF_n^a + F_{n-1}^a \text{ for all } n \geq 2$$

$$\text{if } r \in \left\{ \frac{a+\sqrt{a^2+4}}{2},\ \frac{a-\sqrt{a^2+4}}{2} \right\}.$$

(b) The results of Part (a) imply that $x^2 - ax - 1$ must divide $x^n - xF_n^a - F_{n-1}^a$ for all integers $n \geq 2$ since the only 2 roots of the former polynomial also satisfy the latter polynomial. We now must show that

♦ $$x^n - xF_n^a - F_{n-1}^a = (x^2 - ax - 1)(x^{n-2} + ax^{n-3} + \cdots + xF_{n-2}^a + F_{n-1}^a) \text{ for all integers } n \geq 2 .$$

The case for $n = 2$ is (trivially) true since $F_2^a = a$ and $F_1^a = 1$, which makes both sides of Equation (♦) equal to $x^2 - ax - 1$. For the case $n = 3$, the right side of equation (♦) equals

$$(x^2 - ax - 1)(x + F_2^a) = (x^2 - ax - 1)(x + a) .$$

while the left side is

$$x^3 - xF_3^a - F_2^a = x^3 - x(a^2 + 1) - a .$$

Multiplying out the right side $(x^2 - ax - 1)(x + a)$:

$$\begin{aligned}(x^2 - ax - 1)(x + a) &= x^3 + ax^2 - ax^2 - a^2x - x - a \\ &= x^3 - x(a^2 + 1) - a,\end{aligned}$$

shows the two sides are equal, so we have another base case:

$$x^3 - xF_3^a - F_2^a = (x^2 - ax - 1)(x + F_2^a) .$$

Proceeding inductively, assume that for some positive integer $m > 3$, that equation (♦) is true for $n = m - 1$.

That is,

IA $$x^{m-1} - xF_{m-1}^a - F_{m-2}^a = (x^2 - ax - 1)(x^{m-3} + ax^{m-4} + \cdots + xF_{m-3}^a + F_{m-2}^a)$$

with this Inductive Assumption labeled " IA ". Assuming this equation, it must be shown that

▲ $$\begin{aligned}&x^m - xF_m^a - F_{m-1}^a \\ &\quad = (x^2 - ax - 1)(x^{m-2} + ax^{m-3} + \cdots + x^2F_{m-3}^a + xF_{m-2}^a + F_{m-1}^a)\end{aligned}$$

to complete the induction step.

Note the right side of equation (▲) (yet to be shown true) equals

$$\begin{aligned}&(x^2 - ax - 1)(x^{m-2} + ax^{m-3} + \cdots + x^2F_{m-3}^a + xF_{m-2}^a + F_{m-1}^a) \\ &= (x^2 - ax - 1)(x[x^{m-3} + ax^{m-4} + \cdots + xF_{m-3}^a + F_{m-2}^a] + F_{m-1}^a) .\end{aligned}$$

http://dx.doi.org/10.29322/IJSRP.13.02.2023.p13450

Distributing $(x^2 - ax - 1)$ gives

$$= (x(x^2 - ax - 1)[x^{m-3} + ax^{m-4} + \cdots + xF^a_{m-3} + F^a_{m-2}] + (x^2 - ax - 1)F^a_{m-1})$$
$$= x[(x^2 - ax - 1)(x^{m-3} + ax^{m-4} + \cdots + xF^a_{m-3} + F^a_{m-2})] + (x^2 - ax - 1)F^a_{m-1}$$

Next substituting $x^{m-1} - xF^a_{m-1} - F^a_{m-2}$ from (IA) for the bracketed expression in the previous line provides

$$= x[x^{m-1} - xF^a_{m-1} - F^a_{m-2}] + (x^2 - ax - 1)F^a_{m-1}$$
$$= [x^m - x^2F^a_{m-1} - xF^a_{m-2}] + x^2F^a_{m-1} - axF^a_{m-1} - F^a_{m-1}$$
$$= x^m - x^2F^a_{m-1} - xF^a_{m-2} + x^2F^a_{m-1} - axF^a_{m-1} - F^a_{m-1}$$
$$= x^m - xF^a_{m-2} - axF^a_{m-1} - F^a_{m-1}$$
$$= x^m - x(F^a_{m-2} + aF^a_{m-1}) - F^a_{m-1}$$
$$= x^m - xF^a_m - F^a_{m-1}$$

this last step being true since $F^a_m = F^a_{m-2} + aF^a_{m-1}$ by the recursive rule for F^a_m.
Therefore this shows that

$$x^m - xF^a_m - F^a_{m-1} = (x^2 - ax - 1)(x^{m-2} + ax^{m-3} + \cdots + x^2F^a_{m-3} + xF^a_{m-2} + F^a_{m-1})$$

matching Eqn. (▲) and the statement of Part(b) is proved. ■

This theorem proved may now be applied with $a = 1$ to supply rigorous argument to the validity of Equations (1) and (3) stated before as conjectures. Next we turn our attention in a “negative direction”.

2. Fibonacc–a Integers

If one extends the Fibonacci sequence

$$\mathbf{F} = \{1,1,2,3,5,8,13,\dots\} = \left\{F_n = F_{n-1} + F_{n-2} \mid n \ge 3 \text{ and } F_2 = F_1 = 1\right\}$$

through use of subtraction to one with no lower bound that includes negative integers and $\mathbf{F}$, but still follows the same simple rule well-known for $\mathbf{F}$:

“The next number in the sequence is produced by adding the previous two numbers in the sequence.”

we have

$$\mathbf{G} = \{\dots -21, 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, \dots\}.$$

One might call this $\mathbf{G}$ the *Generalized Fibonacci Sequence*, which has the recursive definition, indexed by $n \in \mathbf{Z}$:

$$\mathbf{G} = \left\{ G_n \mid G_n = \left\{ \begin{array}{ll} F_n, & \text{if } n \ge 1 \\ G_{n+2} - G_{n+1} & \text{if } n \le 0 \end{array} \right. \right\}$$

Then for integer $k \ge 1$, $G_k = F_k$, and $G_0 = 0$, $G_{-1} = 1$, $G_{-2} = -1$, $G_{-3} = 2$, $G_{-4} = -3$,

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$G_{-5} = 5,\ G_{-6} = -8,$ etc.

Can one confirm the following equation

$$\left(\frac{1+\sqrt{5}}{2}\right)^{-5} = 5\left(\frac{1+\sqrt{5}}{2}\right) - 8$$

without getting into the messy mathematics this presents?

Any solution x to the rational equation

$$x^{-5} = 5x - 8$$

would clearly need to be nonzero, and thus would also be a solution to the polynomial equation

$$5x^6 - 8x^5 - 1 = 0$$

which is equivalent to the previous equation for nonzero x.

As shown earlier, both $v_1 = \frac{1+\sqrt{5}}{2}$ and $\overline{v_1} = \frac{1-\sqrt{5}}{2}$ are roots of $x^2 - x - 1$ and polynomial division shows that $\frac{5x^6-8x^5-1}{x^2-x-1} = 5x^4 - 3x^3 + 2x^2 - x + 1$. Multiplication confirms that

$$5x^6 - 8x^5 - 1 = (x^2 - x - 1)(5x^4 - 3x^3 + 2x^2 - x + 1)\ .$$

Therefore this shows that $v_1 = \varphi$ and $\overline{v_1} = \overline{\varphi}$ each satisfy $5x^6 - 8x^5 - 1$ and $x^{-5} = 5x - 8$. So this proves the truth now of both

$$\left(\frac{1+\sqrt{5}}{2}\right)^{-5} = 5\left(\frac{1+\sqrt{5}}{2}\right) - 8 \text{ and } \left(\frac{1-\sqrt{5}}{2}\right)^{-5} = 5\left(\frac{1-\sqrt{5}}{2}\right) - 8$$

without getting into the potential messiness of these last two equations.

Since $G_{-5} = 5$ and $G_{-6} = -8$, we see the more general

(4) $\varphi^n = G_n\varphi + G_{n-1}$, for $n \in \mathbf{Z}$ is true for $n = -5$.

One might speculate further (using positive n to represent negative subscripts) that

(5) $$G_{-n}x^{n+1} + G_{-n-1}x^n - 1 = (x^2 - x - 1)(G_{-n}x^{n-1} + G_{-n+1}x^{n-2} + \cdots - x + 1)$$

(5) also shown true for $n = 5$. Both of these equations (4) and (5) will be shown true in a more general context.

Let us define the <u>*Generalized Fibonacc–a Sequence*</u>:
For any integer $a \geq 1$,
$\mathbf{G}_a$: $\ldots,\ -a^5 - 4a^3 - 3a,\ a^4 + 3a^2 + 1,\ -a^3 - 2a,\ a^2 + 1,\ -a,\ 1,$

$$0,\ 1,\ a,\ a^2 + 1,\ a^3 + 2a,\ a^4 + 3a^2 + 1,\ a^5 + 4a^3 + 3a,\ \ldots$$

whereby $\mathbf{G}_1 = \mathbf{G}$, the Generalized Fibonacci Sequence.

http://dx.doi.org/10.29322/IJSRP.13.02.2023.p13450

Then we may use the notation $G_n^a = F_n^a$ if $n \geq 1$ and

$$\mathbf{G}_a = \left\{ G_n^a \mid G_n^a = \left\{ \begin{array}{ll} F_n^a, & \text{if } n \geq 1 \\ G_{n+2}^a - aG_{n+1}^a & \text{if } n \leq 0 \end{array} \right. \right\}$$

generates further Fibonacc–a Integers.

Example: For $a = 4$, we have

$$\mathbf{G}_4 : \ \ldots, 305, \ -72, \ 17, \ -4, \ 1, \ 0, \ 1, \ 4, \ 17, \ 72, \ 305, \ldots$$

$$v_4 = \frac{a + \sqrt{a^2+4}}{2}\Big|_{a=4} = \frac{4+\sqrt{20}}{2} = 2 + \sqrt{5}$$

and $v_4^{-3} = 17v_4 - 72 \Rightarrow \left(2+\sqrt{5}\right)^{-3} = 17\left(2+\sqrt{5}\right) - 72$, is true since

$\left(2+\sqrt{5}\right)^{-3} = -38 + 17\sqrt{5}$ and $17\left(2+\sqrt{5}\right) - 72 = -38 + 17\sqrt{5}$.

Therefore $v_4 = 2 + \sqrt{5}$ satisfies $x^{-3} = 17x - 72$, which is equivalent to the polynomial equation

$$17x^4 - 72x^3 - 1 = 0 .$$

Therefore the irrational conjugate $\overline{v_4} = 2 - \sqrt{5}$ must also be a solution to this polynomial equation with rational coefficients, which implies that the following

$$(x - v_4)(x - \overline{v_4}) = x^2 - 4x - 1$$

is a factor of $17x^4 - 72x^3 - 1$. The cofactor (or quotient) turns out to be a Fibonacc–a Polynomial as defined earlier since it is a polynomial with Fibonacc–a integer coefficients that are consecutive (within the sequence):

$$17x^2 - 4x + 1 .$$

This is confirmed by the truth of the equation

$$(x^2 - 4x - 1)(17x^2 - 4x + 1) = 17x^4 - 72x^3 - 1$$

What remains is to prove

Theorem 2: Let a be a positive integer and $r \in \left\{ v_a, \ \overline{v_a} \mid v_a = \frac{a+\sqrt{a^2+4}}{2}, \ \overline{v_a} = \frac{a-\sqrt{a^2+4}}{2} \right\}$.

(a) $r^{-1} = r - a$.

(b) $r^n = rG_n^a + G_{n-1}^a$, whenever $n \in \mathbf{Z}$.

(c) For any $n \geq 1$,

$$x^{n+1}G_{-n}^a + x^n G_{-(n+1)}^a - 1 = (x^2 - ax - 1)\left(x^{n-1}G_{-n}^a + x^{n-2}G_{-n+1}^a + \cdots + x^2 G_{-3}^a - ax + 1 \right)$$

http://dx.doi.org/10.29322/IJSRP.13.02.2023.p13450

Examples: For $a = 3$,

$$\mathbf{G}_3 : \ \ldots, -360,\ 109,\ -33,\ 10,\ -3,\ 1,\ 0,\ 1,\ 3,\ 10,\ 33,\ 109,\ 360 \ldots$$

$$v_3 = \frac{a+\sqrt{a^2+4}}{2}\Big|_{a=3} = \frac{3+\sqrt{13}}{2} \quad \text{and}$$

$$v_3^{-5} = 109 v_3 - 360 \Rightarrow \left(\frac{3+\sqrt{13}}{2}\right)^{-5} = 109\left(\frac{3+\sqrt{13}}{2}\right) - 360$$

is true since $\left(\frac{3+\sqrt{13}}{2}\right)^{-5} = -\frac{393}{2} + \frac{109}{2}\sqrt{13}$ and $109\left(\frac{3+\sqrt{13}}{2}\right) - 360 = -\frac{393}{2} + \frac{109}{2}\sqrt{13}$.

The equation

$$x^{-5} = 109x - 360$$

rewritten as an equivalent polynomial equation

$$109x^6 - 360x^5 - 1 = (x^2 - 3x - 1)(109x^4 - 33x^3 + 10x^2 - 3x + 1)$$

factors into Fibonacc–a Polynomials.

We now know the connection between the 2 equations mentioned at the beginning of this article:

$$\left(\frac{3+\sqrt{13}}{2}\right)^{-5} = 109\left(\frac{3+\sqrt{13}}{2}\right) - 360 \quad \text{and}$$

$$109x^6 - 360x^5 - 1 = (x^2 - 3x - 1)(109x^4 - 33x^3 + 10x^2 - 3x + 1) \ .$$

For $a = 5$,

$$\mathbf{G}_5 : \ \ldots, 18901,\ -3640,\ 701,\ -135,\ 26,\ -5,\ 1,\ 0,\ 1,\ 5,\ 26,\ 135,\ 701,\ 3640 \ldots$$

$$\overline{v_5} = \frac{a-\sqrt{a^2+4}}{2}\Big|_{a=5} = \frac{5-\sqrt{29}}{2} \quad \text{and}$$

$$\overline{v_5}^{-6} = -3640\overline{v_5} + 18901 \Rightarrow \left(\frac{5-\sqrt{29}}{2}\right)^{-6} = -3640\left(\frac{5-\sqrt{29}}{2}\right) + 18901 \ ,$$

is true since $\left(\frac{5-\sqrt{29}}{2}\right)^{-6} = -1820\left(5-\sqrt{29}\right) + 18901$ and
$-3640\left(\frac{5-\sqrt{29}}{2}\right) + 18901 = -1820\left(5-\sqrt{29}\right) + 18901$.

http://dx.doi.org/10.29322/IJSRP.13.02.2023.p13450

The rational equation $x^{-6} = -3640x + 18901$ is equivalent to $-3640x^7 + 18901x^6 - 1 = 0$, and

$$-3640x^7 + 18901x^6 - 1 = (x^2 - 5x - 1)(-3640x^5 + 701x^4 - 135x^3 + 26x^2 - 5x + 1)$$

factors into Fibonacc$-a$ Polynomials!

We know now also using methods of the first section that since $\left(\frac{5-\sqrt{29}}{2}\right)^6 = 3640\left(\frac{5-\sqrt{29}}{2}\right) + 701 = 1820\left(5 - \sqrt{29}\right) + 701$, the product

$$\left[-1820\left(5 - \sqrt{29}\right) + 18901\right] \cdot \left[1820\left(5 - \sqrt{29}\right) + 701\right]$$

as suggested at the beginning of this article can only equal 1 without any further calculations since the 1st factor equals $\left(\frac{5-\sqrt{29}}{2}\right)^{-6}$ and the 2nd factor equals $\left(\frac{5-\sqrt{29}}{2}\right)^6$.

Proof of Theorem 2: (a) Since $r^2 = ar + 1$, it follows from dividing both sides of this equation by r that

$$r = a + r^{-1} \Rightarrow r^{-1} = r - a .$$

(b) Th. 1(a) implies the truth of $r^n = rG_n^a + G_{n-1}^a$ for integers $n \geq 2$ since $G_n^a = F_n^a$ when $n \geq 1$.

For $n = 1$, $x^n = xG_n^a + G_{n-1}^a$ is true since for the left-side of the equation, $x^n = x$, and for the right side, $xG_n^a + G_{n-1}^a = x$ since $G_1^a = F_1^a = 1$ and $G_0^a = 0$. Therefore $r^n = rG_n^a + G_{n-1}^a$ for $n = 1$.

For $n = 0$, $x^n = xG_n^a + G_{n-1}^a$ is true since $x^n = x^0 = 1$ and $xG_n^a + G_{n-1}^a = xG_0^a + G_{-1}^a = x \cdot 0 + 1 = 1$ since $G_{-1}^a = 1$. Therefore $r^n = rG_n^a + G_{n-1}^a$ for $n = 0$.

Assume there exists an integer $m < 0$ for which $r^n = rG_n^a + G_{n-1}^a$ is true for $n = m + 1$. Then the inductive assumption

$$r^{m+1} = rG_{m+1}^a + G_m^a$$

is true. Dividing by r yields $r^m = G_{m+1}^a + r^{-1}G_m^a$. Since $G_{m+1}^a = aG_m^a + G_{m-1}^a$ and by Part (a), $r^{-1} = r - a$, it follows that

$$\begin{aligned} r^m &= (aG_m^a + G_{m-1}^a) + r^{-1}G_m^a \\ &= aG_m^a + G_{m-1}^a + (r-a)G_m^a \\ &= aG_m^a + G_{m-1}^a + rG_m^a - aG_m^a \\ &= rG_m^a + G_{m-1}^a \end{aligned}$$

Therefore $r^m = rG_m^a + G_{m-1}^a$. So induction is complete and $r^n = rG_n^a + G_{n-1}^a$, whenever $n \in \mathbf{Z}$.

(c) The equation $x^{n+1}G_{-n}^a + x^nG_{-(n+1)}^a - 1 = (x^2 - ax - 1)\left(x^{n-1}G_{-n}^a + x^{n-2}G_{-n+1}^a + \cdots + x^2G_{-3}^a - ax + 1\right)$ can be shown to be true for $n = 1$.

http://dx.doi.org/10.29322/IJSRP.13.02.2023.p13450

To verify this, since $G^a_{-1} = 1$ and $G^a_{-2} = -a$, it follows that the left side of the proposed equation is

$$\begin{aligned} x^{n+1}G^a_{-n} + x^n G^a_{-(n+1)} - 1 &= x^2 G^a_{-1} + xG^a_{-2} - 1 \\ &= x^2 - ax - 1 \end{aligned}$$

while the right side of this equation is

$$\begin{aligned} (x^2 - ax - 1)\left(G^a_{-n}x^{n-1} + G^a_{-n+1}x^{n-2} + \cdots + G^a_{-3}x^2 - ax + 1 \right) &= (x^2 - ax - 1)G^a_{-1}x^0 \\ &= x^2 - ax - 1 \end{aligned}$$

Therefore $x^{n+1}G^a_{-n} + x^n G^a_{-(n+1)} - 1 = (x^2 - ax - 1)\left(x^{n-1}G^a_{-n} + x^{n-2}G^a_{-n+1} + \cdots + x^2 G^a_{-3} - ax + 1 \right)$ is true for $n = 1$.

Assume now that for some integer $m > 1$ that
$x^{n+1}G^a_{-n} + x^n G^a_{-(n+1)} - 1 = (x^2 - ax - 1)\left(x^{n-1}G^a_{-n} + x^{n-2}G^a_{-n+1} + \cdots + x^2 G^a_{-3} - ax + 1 \right)$ is true for $n = m - 1$.
Then we know that
IA:

$$x^m G^a_{-(m-1)} + x^{m-1}G^a_{-m} - 1 = (x^2 - ax - 1)\left(x^{m-2}G^a_{-(m-1)} + x^{m-3}G^a_{-m+2} + \cdots + x^2 G^a_{-3} - ax + 1 \right)$$

is true.
To complete the induction argument, it must now be shown that

$$x^{m+1}G^a_{-m} + x^m G^a_{-(m+1)} - 1 = (x^2 - ax - 1)\left(x^{m-1}G^a_{-m} + x^{m-2}G^a_{-m+1} + \cdots + x^2 G^a_{-3} - ax + 1 \right)$$

is true.
Since by the recursive definition for $G^a_{-(m-1)}$, we have $G^a_{-m+1} = aG^a_{-m} + G^a_{-m-1}$, if follows that $G^a_{-m-1} = G^a_{-m+1} - aG^a_{-m}$.
Then the left side of the equation to be proved is

$$\begin{aligned} x^{m+1}G^a_{-m} + x^m G^a_{-m-1} - 1 &= x^{m+1}G^a_{-m} + x^m(G^a_{-m+1} - aG^a_{-m}) - 1 \\ &= x^{m+1}G^a_{-m} + x^m G^a_{-m+1} - ax^m G^a_{-m} - 1 \end{aligned}$$

while the right side is

$$\begin{aligned} &(x^2 - ax - 1)\left(x^{m-1}G^a_{-m} + x^{m-2}G^a_{-m+1} + \cdots + x^2 G^a_{-3} - ax + 1 \right) \\ &= ((x^2 - ax - 1)x^{m-1}G^a_{-m} + [(x^2 - ax - 1)(x^{m-2}G^a_{-m+1} + x^{m-3}G^a_{-m+2} + \cdots + x^2 G^a_{-3} - ax + 1)]) \end{aligned}$$

This last equation results from distribution of $x^2 - ax - 1$.

Now substituting in $x^m G^a_{-(m-1)} + x^{m-1}G^a_{-m} - 1$ into the bracketed expression of this last equation via the inductive assumption (IA) yields

$$\begin{aligned} &((x^2 - ax - 1)x^{m-1}G^a_{-m} + [x^m G^a_{-m+1} + x^{m-1}G^a_{-m} - 1]) \\ &= x^{m+1}G^a_{-m} - ax^m G^a_{-m} - x^{m-1}G^a_{-m} + x^m G^a_{-m+1} + x^{m-1}G^a_{-m} - 1 \\ &= x^{m+1}G^a_{-m} - ax^m G^a_{-m} + x^m G^a_{-m+1} - 1 \end{aligned}$$

This last expression equals what was found for the left side of the proposed equation.

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Therefore

$$x^{m+1}G^a_{-m} + x^m G^a_{-(m+1)} - 1 = (x^2 - ax - 1)\Big(x^{m-1}G^a_{-m} + x^{m-2}G^a_{-m+1} + \cdots + x^2 G^a_{-3} - ax + 1\Big)$$

and induction is complete. ■

3. Solvable Quintics, Sextics

Theora 1 and 2 of this paper provide factorizations for special cases of polynomials of the form $ax^n + bx + c$ and $ax^n + bx^{n-1} + c$ in which a, b, c are nonzero complex numbers. These factorizations involve polynomial factors of degrees 2 and $n-2$. For degree $n = 5$ or $n = 6$, these factorizations result in quintic and sextic polynomials solvable by radicals since the degrees of the factors are all less than 5. This provides a different method, or viewpoint, than that discussed in [1] or [5].

Corollary 3: For any integer $a \geq 1$, $x^5 - xF^a_5 - F^a_4 = 0$ and $x^5 G^a_{-4} + x^4 G^a_{-5} - 1 = 0$ represent 2 classes of solvable quintic polynomial equations, and $x^6 - xF^a_6 - F^a_5 = 0$ and $x^6 G^a_{-5} + x^5 G^a_{-6} - 1 = 0$ represent 2 classes of solvable sextic polynomial equations. To view these polynomials more explicitly in terms of a, we have for the quintics:

$$x^5 - x(a^4 + 3a^2 + 1) - (a^3 + 2a) = 0 \text{ and } x^5(-a^3 - 2a) + x^4(a^4 + 3a^2 + 1) - 1 = 0$$

and for the sextics:

$$x^6 - x(a^5 + 4a^3 + 3a) - (a^4 + 3a^2 + 1) = 0 \text{ and } x^6(a^4 + 3a^2 + 1) + x^5(-a^5 - 4a^3 - 3a) - 1 = 0$$

Examples: (a) For $a = 4$, consider the sequence $1, 4, 17, 72, 305$. Then by applying Th. 1, $x^5 - 305x - 72 = (x^2 - 4x - 1)(x^3 + 4x^2 + 17x + 72)$ is solvable by radicals since any polynomial of degree 4 or less is solvable by radicals.
(b) For $a = 5$, consider the sequence $701, -135, 26, -5, 1, 0$. Then by Th. 2 $-135x^5 + 701x^4 - 1 = (x^2 - 5x - 1)(-135x^3 + 26x^2 - 5x + 1)$ is solvable by radicals for the same reason.

Proof of Cor. 3: Theorem 1 with $n = 5$, implies that $x^5 - xF^a_5 - F^a_4 = (x^2 - ax - 1)(x^3 + ax^2 + xF^a_3 + F^a_4)$, so since any polynomial equation of degree 4 or less is solvable by radicals, then so is $x^5 - xF^a_5 - F^a_4 = 0$ for all integers $a \geq 1$. Theorem 2 with $n = 4$, implies that $x^5 G^a_{-4} + x^4 G^a_{-5} - 1 = (x^2 - ax - 1)(x^3 G^a_{-4} + x^2 G^a_{-3} - ax + 1)$, so $x^5 G^a_{-4} + x^4 G^a_{-5} - 1 = 0$ is also solvable by radicals for all integers $a \geq 1$ This provides two classes of quintic polynomials that factor into solvable polynomials.

Furthermore, with $n = 6$, Th. 1 provides $x^6 - xF^a_6 - F^a_5 = (x^2 - ax - 1)(x^4 + ax^3 + x^2 F^a_3 + xF^a_4 + F^a_5)$. Then $n = 5$ with Th. 2 provides $x^6 G^a_{-5} + x^5 G^a_{-6} - 1 = (x^2 - ax - 1)\Big(x^4 G^a_{-5} + x^3 G^a_{-4} + x^2 G^a_{-3} - ax + 1\Big)$, giving 2 classes of sextic polynomials that factor into solvable polynomials. ■

Note: In [4] it is proved that the method of apolar invariants, in particular while utilizing the $A_{3,5}$ – apolar invariant defined in [3], results in no nonzero cubic polynomial apolar to either quintic of the form $ax^5 + bx + c$ nor $ax^5 + bx^4 + c$, where a, b, and c are complex numbers. This implies that methods of apolar invariants break down in these cases. Cor. 3

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now provides methods towards solving quintic polynomials in these special cases for which the method of apolar invariants has limitations.

Now some questions.

Questions 1: Can the classes of polynomials treated in this paper be expanded by extending the defining parameters in the following ways?

1(a): Extend beyond the integers $a \geq 1$ to include a equal to a nonzero real number, for which v_a would still be a real number.

1(b): Extend beyond real number values for a to include complex numbers for which v_a would not necessarily be real.

1(c): Extend to include additional parameters for which the sequence used here, $1,\ a,\ a^2+1,\ \ldots$ becomes, for example, $b,\ a,\ a^2+b,\ \ldots$.

Question 2: What patterns emerge by applying methods of apolar invariants to solving Fibonacc $-a$ integer polynomial equations, such as $x^3+3x^2+10x+33=0$ or $109x^4-33x^3+10x^2-3x+1=0$?

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