Analytical Analysis of the Dynamic Buckling Load of a Geometrical Imperfect Column Lying on a Nonlinear Elastic Foundation Trapped By a Time Dependent Load

Onuoha N.O.
Department of Mathematics, Imo State Universtiy, Owerri, Nigeria

DOI: 10.29322/IJSRP.13.07.2023.p13906
http://dx.doi.org/10.29322/IJSRP.13.07.2023.p13906

Paper Received Date: 14th May 2023
Paper Acceptance Date: 23rd June 2023
Paper Publication Date: 6th July 2023

Abstract
Dynamic stability and dynamic buckling loads of elastic structures have gained numerous interest in recent research. It is because of its wide area of application in many fields of engineering such as civil engineering, architectural engineering and nuclear power plant engineering. Dynamic buckling load of a geometrically imperfect column lying on a nonlinear elastic foundation trapped by a time dependent load is discussed here. Regular perturbation method and asymptotic expansions are applied to solve the differential equation. The effect of damping and geometric imperfections on the dynamic buckling load at different modes are x-rayed. The result shows that increase in geometric imperfections decreases the dynamic buckling load and that damping alters the vibration amplitude of the dynamic buckling load. For the mode, as it increases, the dynamic buckling load reduces.

Keywords: Dynamic stability, Dynamic buckling load, Geometric imperfections, Elastic structures, Vibration amplitude, Regular perturbation, Asymptotic expansion, Damping, Dominant modes,

1. Introduction

Research on the dynamic buckling analysis of structures has gained much interest due to its application in analyzing the safety and design conditions of thin- walled elastic structures such as those used in air-craft manufacturing, aero-space vehicles and other capital- intensive aeronautical and astronautical but imperfection-sensitive machines. Dynamic buckling of structures is of utmost engineering importance, for it is pertinent to investigate at what level of dynamic loading a certain elastic material will sustain a particular dynamic loading system before the material becomes unstable and what should be the contributing factors to early instability of the structure. Adhikari and Woodhouse [1] in their research noted that the study of damping forces in a vibrating structure has long been an active area of research in structural dynamics as demanded by modern engineering practice.
The genesis of dynamic buckling can be traced to a series of research originally done by Budiansky and Hutchinson [3,4,5], and Hutchinson and Budiansky [12]. Over the years, reasonable improvement have been made and the subject matter has grown in leaps and bounds to attract contributions from other researchers which include Svalbonas and Kalnins [22], Huyan and Simitses [14], Simitses [20] and Danielson [6] among others. However, most of these early investigations did not incorporate viscous damping in their investigations. In this regard therefore, Ette [7,8,9], incorporated damping in his research. The dynamic buckling was similarly investigated by Sapsis et al [18] while, Karagiozova [13] revisited the dynamic plastic and dynamic progressive buckling of elastic-plastic circular shells. Simitses [19], extended dynamic buckling analysis to discuss static preloading on the dynamic stability of structures. The investigation by Lindberg [15] is limited to simple linear systems, including bars, and has a section on dynamic buckling with random imperfections and experiments. Ette and Udo-Akpan [11] investigated the dynamic buckling of a model structure with quadratic non linearity struck by a step load.
Al-khazraji et al [2] studied on the improvement of dynamic buckling behaviour of intermediate aluminized stainless steel columns. Aghdam and Schroeder [16] studied the dynamic buckling of crash boxes under an impact load. In the study of buckling, Song-Hak et al [21] investigated the dynamic buckling of composite structures subjected to impulse loads using the Lyapunov Exponent. Researchers in engineering mainly used numerical methods to solve and analyze the dynamic stability of structures irrespective of the loading system. In this research, multiple two-timing regular perturbation and asymptotic expansions are employed.

2. FORMULATION OF THE PROBLEM

In this research, we let $Z(X,T)$ be normal displacement deemed continuous in the space variable X and time T. The column is considered viscously damped with damping coefficient Q, where the damping is proportional to the velocity. The column is trapped by a load $G(T)$ and the relevant differential equation (Ette and Osuji, [10]) is given by

$$m_o Z_{,TT} + QZ_{,T} + EIZ_{,XXXX} + 2G(T)Z_{,XX} + \left(k_1 - k_3 Z^2\right)Z = -2G(T)\frac{d^2\bar{Z}}{dX^2} \quad (1a) \qquad T > 0,\ 0 < X < \pi,$$

$$Z = Z_{,XX} = 0, \text{ at } X = 0, \pi \quad (1b)$$

$$Z(X,0) = Z_{,T}(X,0) = 0,\ 0 < X < \pi \quad (1c)$$

Here, a subscript following a comma indicates partial differentiation. m_o is the mass per unit length of the column, EI is the bending stiffness, where E is the Young's modulus and I is the moment of inertia, and $k_1 > 0,\ k_3 > 0$ are constants. The column rests on a nonlinear elastic foundation that exerts a force per unit length of $\left(k_1 - k_3 Z^2\right)Z$. $\bar{Z}(X)$ is a twice-differentiable stress-free imperfection.

3. Nondimensionalized Quantities

$$x = \left(\frac{l_1}{EI}\right)^{\frac{1}{4}} X,\ z = \left(\frac{l_3}{l_1}\right)^{\frac{1}{2}} Z,\ \lambda f(t) = \frac{G(T)}{2\left(EIl_1\right)^{\frac{1}{2}}},\ t = \left(\frac{l_1}{m_0}\right)^{\frac{1}{2}} T,\ \varepsilon\bar{z} = \left(\frac{l_3}{l_1}\right)^{\frac{1}{2}} \bar{Z},$$

$$2\delta = \frac{Q}{\left(m_0 l_1\right)^{\frac{1}{2}}}$$

Hence, the original differential equation (1a), together with the boundary conditions and initial conditions becomes

$$z_{,\bar{t}\bar{t}} + 2\delta z_{,\bar{t}} + z_{,xxxx} + 2\lambda f(\bar{t}) z_{,xx} + \left(1 - z^2\right)z = 2\lambda\varepsilon f(\bar{t})\frac{d^2\bar{z}}{dx^2} \qquad \bar{t} > 0, 0 < x < \pi \quad (2a)$$

$$z = z_{,xx} = 0 \ at\ x = 0, \pi \quad (2b)$$

$$z(x,0) = z_{,\bar{t}}(x,0) = 0,\ 0 < x < \pi \quad (2c)$$

Here, δ and ε are two small but mathematically distinct and independent parameters upon which a two-timing regular perturbation procedure will be initiated by means of regular perturbation. We are to determine the dynamic buckling load λ_D on which the column buckles dynamically.

4. Regular Perturbation Solution of the dynamic problem

Let

$$\tau = \delta\bar{t}$$

$$\frac{\partial t}{\partial\bar{t}} = \left(m^4 - 2m^2\lambda + 1\right)^{\frac{1}{2}} = \varphi_m^{\frac{1}{2}} \quad (3)$$

$$t = \bar{t} + \frac{1}{\delta}\left(\mu_2\varepsilon^2 + \mu_3\varepsilon^3 + \ldots\right),\ \mu_i = \mu_i(\tau);\ \mu_i(0) = 0, i = 1, 2, 3, \ldots$$

Further let

$$z(\bar{t}, x) = U(x, t, \tau) \quad (4)$$

It follows that

$$z_{,\bar{t}} = \left(m^4 - 2m^2\lambda + 1\right)^{\frac{1}{2}} U_{,t} + \left(\mu_2'\varepsilon^2 + \mu_3'\varepsilon^3 + \ldots\right)U_{,t} + \delta U_{,\tau} \quad (5)$$

$$z_{,\bar{t}\bar{t}} = \left(m^4 - 2m^2\lambda + 1\right)U_{,tt} + \left(\mu_2'\varepsilon^2 + \mu_3'\varepsilon^3 + \ldots\right)^2 U_{,tt} + \delta^2 U_{,\tau\tau} +$$
$$2\left(m^4 - 2m^2\lambda + 1\right)^{\frac{1}{2}}\left(\mu_2'\varepsilon^2 + \mu_3'\varepsilon^3 + \ldots\right)U_{,tt} + 2\delta\left(\mu_2'\varepsilon^2 + \mu_3'\varepsilon^3 + \ldots\right)U_{,t\tau} + \quad (6)$$
$$2\delta\left(m^4 - 2m^2\lambda + 1\right)^{\frac{1}{2}} U_{,t\tau} + \delta\left(\mu_2''\varepsilon^2 + \mu_3''\varepsilon^3 + \ldots\right)U_{,t}$$

5. Asymptotic Series Assumption

Let

$$U(x,t,\tau)=\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}U^{ij}(x,t,\tau)\varepsilon^i\delta^j \tag{7}$$

Substituting (5), (6) and (7) into (2a) yields $(\varepsilon.1): U^{(10)}_{,tt}+\varphi_m^{-1}\left(U^{(10)}_{,xxxx}+2\lambda U^{(10)}_{,xx}+U^{(10)}\right)=-2\lambda\varphi_m^{-1}\bar{z}_{xx}$ (8) $\varphi_m=\left(m^4-2m^2\lambda+1\right)$ (9)

$$(\varepsilon.\delta): U^{(11)}_{,tt}+\varphi_m^{-1}\left(U^{(11)}_{,xxxx}+2\lambda U^{(11)}_{,xx}+U^{(11)}\right)=-2\varphi_m^{-1}\left(U^{(10)}_{,t\tau}+U^{(10)}_{,t}\right) \tag{10}$$

$$(\varepsilon.\delta^2): U^{(12)}_{,tt}+\varphi_m^{-1}\left(U^{(12)}_{,xxxx}+2\lambda U^{(12)}_{,xx}+U^{(12)}\right)=-2\varphi_m^{-1}\left\{\left(U^{(11)}_{,t\tau}+U^{(11)}_{,t}\right)+\left(U^{(10)}_{,\tau\tau}+U^{(10)}_{,\tau}\right)\right\} \tag{11}$$

$$(\varepsilon^2.1): U^{(20)}_{,tt}+\varphi_m^{-1}\left(U^{(20)}_{,xxxx}+2\lambda U^{(20)}_{,xx}+U^{(20)}\right)=0 \tag{12}$$

$$(\varepsilon^2.\delta): U^{(21)}_{,tt}+\varphi_m^{-1}\left(U^{(21)}_{,xxxx}+2\lambda U^{(21)}_{,xx}+U^{(21)}\right)=0 \tag{13}$$

$$(\varepsilon^2.\delta^2): U^{(22)}_{,tt}+\varphi_m^{-1}\left(U^{(22)}_{,xxxx}+2\lambda U^{(22)}_{,xx}+U^{(22)}\right)=-2\varphi_m^{-1}\left(U^{(21)}_{,t\tau}+U^{(21)}_{,t}\right) \tag{14}$$

$$(\varepsilon^3.1): U^{(30)}_{,tt}+\varphi_m^{-1}\left(U^{(30)}_{,xxxx}+2\lambda U^{(30)}_{,xx}+U^{(30)}\right)=\varphi_m^{-1}\eta\left(U^{(10)}\right)^3-2\varphi_m^{-\frac{1}{2}}\mu_2' U^{(10)}_{,tt} \tag{15}$$

$$(\varepsilon^3.\delta): U^{(31)}_{,tt}+\varphi_m^{-1}\left(U^{(31)}_{,xxxx}+2\lambda U^{(31)}_{,xx}+U^{(31)}\right)=3\eta\varphi_m^{-1}\left(U^{(10)}\right)^2U^{(11)}-2\varphi_m^{-\frac{1}{2}}\mu_2' U^{(11)}_{,tt}-2\varphi_m^{-1}\mu_2' U^{(10)}_{,t\tau}-2\varphi_m^{-\frac{1}{2}}U^{(30)}_{,t\tau}-\varphi_m^{-1}\mu_2'' U^{(10)}_{,t}-2\varphi_m^{-\frac{1}{2}}U^{(30)}_{,t}-2\varphi_m^{-1}\mu_2' U^{(10)}_{,t}$$

(16)

$$(\varepsilon^3.\delta^2): U^{(32)}_{,tt}+\varphi_m^{-1}\left(U^{(32)}_{,xxxx}+2\lambda U^{(32)}_{,xx}+U^{(32)}\right)=3\eta\varphi_m^{-1}\left\{\left(U^{(10)}\right)^2U^{(12)}+U^{(10)}\left(U^{(11)}\right)^2\right\}-$$
$$-2\varphi_m^{-\frac{1}{2}}\mu_2' U^{(12)}_{,tt}-\varphi_m^{-1}\mu_2' U^{(11)}_{,t\tau}-2\varphi_m^{-\frac{1}{2}}U^{(31)}_{,t\tau}-\varphi_m^{-1}\mu_2'' U^{(11)}_{,t}-2\varphi_m^{-\frac{1}{2}}U^{(31)}_{,t}-2$$
$$2\varphi_m^{-1}\mu_2' U^{(11)}_{,t}-\varphi_m^{-1}U^{(30)}_{,\tau\tau}-2\varphi_m^{-1}U^{(30)}_{,\tau} \tag{17}$$

The associated initial conditions evaluated at $(t,\tau)=(0,0)$ using (2c)

6. DISPLACEMENT MODES $U^{(ij)}$

Let

$$U^{(ij)}=U^{(ij)}(x,t,\tau)=\sum_{n=1}^{\infty}U_n^{(ij)}(t,\tau)\sin nx \tag{18}$$

where we have assumed

$$\bar{z}=\bar{a}_m\sin mx,\ \ |\bar{a}_m|<<1, m=1,2,3,... \tag{19}$$

Solutions of equation ε

Substitute (18) and (19) into (13)

$$\sum_{n=1}^{\infty}\left[U^{(10)}_{n,tt}+\left(\frac{\varphi_n}{\varphi_m}\right)U_n^{(10)}\right]\sin nx=\frac{2\lambda m^2\bar{a}_m\sin mx}{\varphi_m} \tag{20}$$

where

$$\varphi_n=\left(n^4-2n^2\lambda+1\right) \tag{21}$$

Thus, for n = m

$$U^{(10)}_{m,tt}+U^{(10)}_m=\frac{2\lambda m^2\bar{a}_m}{\varphi_m}=B_m\bar{a}_m \tag{22}$$

where

$$B_m = \frac{2\lambda m^2}{\varphi_m} \qquad (23)$$

$$U_m^{(10)}(0,0)=0,\ U_{m,t}^{(10)}(0,0)=0 \qquad (24)$$

The solutions of (22) – (24) are

$$U_m^{(10)}(t,\tau)=\alpha_m^{(10)}(\tau)\cos t+\beta_m^{(10)}(\tau)\sin t+B_m\bar{a}_m \qquad (25)$$

$$\beta_m^{(10)}(0)=0 \qquad (26a)$$

$$\alpha_m^{(10)}(0)=-B_m\bar{a}_m \qquad (26b)$$

Solutions of equation $\varepsilon.\delta$

Substitute equations (28) and (29) into (14)

$$\sum_{n=1}^{\infty}\left[U_n^{(11)}+\left(\frac{\varphi_n}{\varphi_m}\right)U_n^{(11)}\right]\sin nx=-2\varphi_m^{-1}\left(U_{m,t\tau}^{(10)}+U_{m,t}^{(10)}\right)\sin mx \qquad (27)$$

For $n=m$, equation (27) becomes

$$U_{m,tt}^{(11)}+U_m^{(11)}=-2\varphi_m^{-1}\left(U_{m,t\tau}^{(10)}+U_{m,t}^{(10)}\right) \qquad (28)$$

$$U_m^{(11)}(0,0)=0,\ U_{m,t}^{(11)}+\varphi_m^{-\frac{1}{2}}U_{m,\tau}^{(10)}=0 \qquad (29)$$

Substitute the known term $U_m^{(10)}$ on the right hand side of equation

$$U_{m,tt}^{(11)}+U_m^{(11)}=-2\varphi_m^{-1}\left[\left\{-\alpha_m^{(10)\prime}(\tau)-\alpha_m^{(10)}(\tau)\right\}\sin t+\left\{\beta_m^{(10)\prime}(\tau)+\beta_m^{(10)}(\tau)\right\}\cos t\right]$$

(30)

To ensure a bounded solution in t, coefficients of $\cos t$ and $\sin t$ in (30) are equated to zero respectively and the resulting solutions are

$\beta_m^{(10)}(\tau)=0$ (31a) $\alpha_m^{(10)}(\tau)=-B_m\bar{a}_m e^{-\tau}$

(31b)

Solving the remaining part of (30), and (29)

$$U_m^{(11)}(t,\tau)=\alpha_m^{(11)}(\tau)\cos t+\beta_m^{(11)}(\tau)\sin t \qquad (32)$$

$$\alpha_m^{(11)}(0)=0 \qquad (33a)$$

$$\beta_m^{(11)}(0)=-\varphi_m^{-\frac{1}{2}}B_m\bar{a}_m \qquad (33b)$$

Solutions of equation $\varepsilon.\delta^2$

Substitute (28) and (29) into (15)

$$\sum_{n=1}^{\infty}\left[U_{n,tt}^{(12)}+\frac{\varphi_n}{\varphi_m}U_n^{(12)}\right]\sin nx=-2\varphi_m^{-1}\left(U_{m,t\tau}^{(11)}+U_{m,t}^{(11)}\right)\sin mx-2\varphi_m^{-1}\left(U_{,\tau}^{(10)}+U_{,\tau\tau}^{(10)}\right)\sin mx$$

(34)

For $n=m$, equation (34) becomes

$$U_{m,tt}^{(12)}+U_m^{(12)}=-2\varphi_m^{-1}\left\{\left(U_{m,t\tau}^{(11)}+U_{m,t}^{(11)}\right)+\left(U_{m,\tau}^{(10)}-U_{m,\tau\tau}^{(10)}\right)\right\} \qquad (35$$

$$U_m^{(12)}(0,0)=0,\ U_{m,t}^{(12)}(0,0)+\varphi_m^{-\frac{1}{2}}U_{m,\tau}^{(11)}(0,0) \qquad (36)$$

Substituting for $U_m^{(10)}$ and $U_m^{(11)}$ into (36), gives

$$U_{m,tt}^{(12)}+U_m^{(12)}=-2\varphi_m^{-1}\left[-\left(\alpha_m^{(11)\prime}+\alpha_m^{(11)}\right)\sin t+\left\{\left(\beta_m^{(11)\prime}+\beta_m^{(11)}\right)+\left(\alpha_m^{(10)\prime\prime}+\alpha_m^{(10)\prime}\right)\right\}\cos t\right]$$

(37)

To ensure a bounded solution in t, coefficients of $\cos t$ and $\sin t$ in (37) are equated to zero respectively and the resulting solutions are

$$\beta_m^{(11)}(\tau) = -\varphi_m^{-\frac{1}{2}} e^{-\tau} B_m \bar{a}_m \quad (38a) \quad \alpha_m^{(11)}(\tau) = 0$$

(38b)

Solving the remaining part of (37), and (36)

$$U_m^{(12)}(t,\tau) = \alpha_m^{(12)}(\tau)\cos t + \beta_m^{(12)}(\tau)\sin t \quad (39) \quad \beta_m^{(12)}(0) = 0$$

(40a)

$$\alpha_m^{(12)}(0) = 0 \quad (40b)$$

Solutions of equation ε^2

Substitute (28) and (29) into (16)

$$\sum_{n=1}^{\infty}\left[U_{n,tt}^{(20)} + \frac{\varphi_n}{\varphi_m} U_n^{(20)}\right]\sin nx = 0$$

(41)

For $n = m$, (41) becomes

$$U_{m,tt}^{(20)} + U_m^{(20)} = 0 \quad (42)$$

$$U^{(20)}(0,0) = 0, \;\; U_{m,t}^{(20)}(0,0) = 0 \quad (43)$$

Solving (42) and (43)

$$U_m^{(20)}(t,\tau) = \alpha_m^{(20)}(\tau)\cos t + \beta_m^{(20)}(\tau)\sin t \quad (44)$$

$$\beta_m^{(20)} = 0 \quad (45a)$$

$$\alpha_m^{(20)}(0) = 0 \quad (45b)$$

Solutions of equation $\varepsilon^2.\delta$

Substitute (28) and (29) into (16)

$$\sum_{n=1}^{\infty}\left[U_{n,tt}^{(21)} + \left(\frac{\varphi_n}{\varphi_m}\right) U_n^{(21)}\right]\sin nx = 0 \quad (46)$$

For $n = m$, equation (46) becomes

$$U_{m,tt}^{(21)} + U_m^{(21)} = 0 \quad (47)$$

$$U_m^{(21)}(0,0) = 0, U_{m,t}^{(21)}(0,0) = 0 \quad (48)$$

Solving (47) and (48),

$$U_m^{(21)}(t,\tau) = \alpha_m^{(21)}(\tau)\cos t + \beta_m^{(21)}(\tau)\sin t \quad (49)$$

$$\beta_m^{(21)}(0) = 0 \quad (50a)$$

$$\alpha_m^{(21)}(0) = 0 \quad (50b)$$

Solutions of equation $\varepsilon^2.\delta^2$

Substitute (28) and (29) into (17)

$$\sum_{n=1}^{\infty}\left[U_n^{(22)} + \left(\frac{\varphi_n}{\varphi_m}\right) U_n^{(22)}\right]\sin nx = -2\varphi_m^{-1}\left(U_{m,t\tau}^{(21)} + U_{m,t}^{(21)}\right)\sin mx \quad (51)$$

For $n = m$, (51) becomes

$$U_{m,tt}^{(22)} + U_m^{(22)} = -2\varphi_m^{-1}\left(U_{m,t\tau}^{(21)} + U_{m,t}^{(21)}\right) \quad (52)$$

$$U_m^{(22)}(0,0) = 0, \;\; U_{m,t}^{(22)} + \varphi_m^{-\frac{1}{2}} U_{m,\tau}^{(22)} = 0 \quad (53)$$

Substituting for $U_m^{(21)}$ into (52), gives

$$U_{m,tt}^{(22)} + U_m^{(22)} = 0 \quad (54)$$

Solving equations (54) and (53)

$$U_m^{(22)}(t,\tau) = \alpha_m^{(22)}(\tau)\cos t + \beta_m^{(22)}(\tau)\sin t \quad (55)$$

$$\beta_m^{(22)}(0) = 0 \quad (56a)$$

$$\alpha_m^{(22)}(0) = 0 \quad (56b)$$

Solutions of equation ε^3

$$\sum_{n=1}^{\infty}\left[U_{n,tt}^{(30)}+\frac{\varphi_n}{\varphi_m}U_n^{(30)}\right]\sin nx=\left[\eta\varphi_m^{-1}\left(U_m^{(10)}\right)^3-2\varphi_m^{-\frac{1}{2}}\mu_2' U_{m,tt}^{(10)}\right]\sin mx \quad (57)$$

Substituting for $\left(U_m^{(10)}\right)^3$ and $U_{m,tt}^{(10)}$ into (57), gives

$$\sum_{n=1}^{\infty}\left[U_{n,tt}^{(30)}+\left(\frac{\varphi_n}{\varphi_m}\right)U_n^{(30)}\right]\sin nx=\frac{1}{4}(3\sin mx-\sin 3mx)\left[r_0+r_1\cos t+r_2\cos 2t+r_3\cos 3t\right]+2\varphi_m^{-\frac{1}{2}}\mu_2'(\tau)\left(\alpha_m^{(10)}(\tau)\cos t\right)\sin mx$$

(58)
where

$$r_0(\tau)=\frac{3}{4}\varphi_m^{-1}\left(B_m\bar{a}_m\right)^3\left\{3e^{-2\tau}+1\right\} \quad (59a)$$

$$r_2(\tau)=-\frac{9}{8}\varphi_m^{-1}e^{-2\tau}\left(B_m\bar{a}_m\right)^3 \quad (59b)$$

$$r_3(\tau)=\frac{3}{16}\varphi_m^{-1}e^{3\tau}\left(B_m\bar{a}_m\right)^3 \quad (59c)$$

For $n=m$, equation (58) becomes

$$U_{m,tt}^{(30)}+U_m^{(30)}=\frac{3}{4}\left[r_0+r_1\cos t+r_2\cos 2t+r_3\cos 3t\right]+2\varphi_m^{-\frac{1}{2}}\mu_2'(\tau)\alpha_m^{(10)}(\tau)\cos t \quad (60)$$

$$U_{m,tt}^{(30)}(0,0)=0,\ U_{m,t}^{(30)}(0,0)+\varphi_m^{-\frac{1}{2}}\mu_2'(0)U_{m,t}^{(10)}(0)=0 \quad (61)$$

To ensure a bounded solution in t, coefficients of $\cos t$ in (60) is equated to zero and the resulting solution is

$$\mu_2'(\tau)=-\frac{3}{8\alpha_m^{(10)}(\tau)}r_1\varphi_m^{-\frac{1}{2}} \quad (62a)$$

where

$$\mu_2'(0)=-\frac{45}{32}\varphi_m^{-\frac{1}{2}}\left(B_m\bar{a}_m\right)^2 \quad (62b)$$

Solving the remaining part of (60), and (61)

$$U_m^{(30)}(t,\tau)=\alpha_m^{(30)}(\tau)\cos t+\beta_m^{(30)}(\tau)\sin t+\frac{3}{4}\eta\varphi_m^{-1}\left\{r_0(\tau)-\frac{1}{3}r_2(\tau)\cos 2t-\frac{1}{8}r_3(\tau)\cos 3t\right\} \quad (63)$$

$$\beta_m^{(30)}(0)=0 \quad (64a)$$

$$\alpha_m^{(30)}(0)=-\frac{195}{128}\varphi_m^{-1}\left(B_m\bar{a}_m\right)^3 \quad (64b)$$

For $n=3m$, equation (58) becomes

$$U_{3m,tt}^{(30)}+\Omega_m^2U_{3m}^{(30)}=-\frac{1}{4}\left(r_0+r_1\cos t+r_2\cos 2t+r_3\cos 3t\right) \quad (65)$$

where

$$\Omega_m^2=\frac{\varphi_{3m}}{\varphi_m},\ \varphi_{3m}=\left(81m^4-18m^2\lambda+1\right)>0,\forall m=1,2,3,\ldots \quad (66)$$

with the initial conditions

$$U_{3m}^{(30)}(0,0)=0,\ U_{3m,t}^{(30)}(0)+\varphi_m^{-\frac{1}{2}}\mu_2'(0)U_{m,t}^{(10)}(0)=0 \quad (67)$$

On solving (65), and (66),

$$U_{3m}^{(30)}\left(t,\tau\right)=\alpha_{3m}^{(30)}\left(\tau\right)\cos\Omega_m t+\beta_{3m}^{(30)}\left(\tau\right)\sin\Omega_m t-\frac{1}{4}\varphi_m^{-1}\left\{\frac{r_0\left(\tau\right)}{\Omega_m^2}+\frac{r_1\left(\tau\right)\cos t}{\Omega_m^2-1}+\frac{r_2\left(\tau\right)\cos 2t}{\Omega_m^2-4}+\frac{r_3\left(\tau\right)\cos 3t}{\Omega_m^2-9}\right\}$$

(68)

$$\beta_{3m}^{(30)}\left(0\right)=0 \quad (69a)$$

$$\alpha_{3m}^{(30)}\left(0\right)=\frac{1}{4}\varphi_m^{-1}\left(B_m\bar{a}_m\right)^3\left\{\frac{5}{2\Omega_m^2}-\frac{15}{4\left(\Omega_m^2-1\right)}+\frac{3}{2\left(\Omega_m^2-4\right)}-\frac{1}{4\left(\Omega_m^2-9\right)}\right\} \quad (69b)$$

From (18),

$$U^{(30)}=U_m^{(30)}\sin mx+U_{3m}^{(30)}\sin 3mx \quad (70)$$

Solutions of equation $\varepsilon^3.\delta$

$$\sum_{n=1}^{\infty}\left(U_{n,tt}^{(31)}+\frac{\varphi_n}{\varphi_m}U_n^{(31)}\right)\sin nx=\frac{3}{4}\left(U_m^{(10)}\right)^2U_m^{(11)}\left(3\sin mx-\sin 3mx\right)-2\varphi_m^{-\frac{1}{2}}\mu_2'\left(\tau\right)U_{m,tt}^{(11)}\sin mx-2\mu_2'\varphi_m^{-1}U_{m,t\tau}^{(10)}\sin mx-2\varphi_m^{-\frac{1}{2}}\left(U_{m,t\tau}^{(30)}\sin mx+U_{3m,t\tau}^{(30)}\sin 3mx\right)-\mu_2''\varphi_m^{-1}U_{m,t}^{(10)}\sin mx-2\varphi_m^{-\frac{1}{2}}\left(U_{m,t}^{(30)}\sin mx+U_{3m,t}^{(30)}\sin 3mx\right)-2\varphi_m^{-1}\mu_2'\left(U_{m,t}^{(10)}\sin mx\right) \quad (71)$$

$$\left(U_m^{(10)}\right)^2U_m^{(11)}=\left\{\frac{1}{2}\left(\alpha_m^{(10)}\left(\tau\right)\right)^2+\left(B_m\bar{a}_m\right)^2+2B_m\bar{a}_m\alpha_m^{(10)}\left(\tau\right)\cos t+\frac{1}{2}\left(\alpha_m^{(10)}\left(\tau\right)\right)^2\cos t\right\}\beta_m^{(11)}\left(\tau\right)\sin t \quad (72a)$$

$$\left(U_m^{(10)}\right)^2U_m^{(11)}=r_4\sin t+\frac{1}{2}r_5\sin 2t+\frac{1}{2}r_6\left(\sin 3t-\sin t\right) \quad (72b)$$

where

$$r_4\left(\tau\right)=\beta_m^{(11)}\left(\tau\right)\left\{\frac{1}{2}\left(\alpha_m^{(10)}\left(\tau\right)\right)^2+\left(B_m\bar{a}_m\right)^2\right\} \quad (73a)$$

$$r_5\left(\tau\right)=2B_m\bar{a}_m\alpha_m^{(10)}\left(\tau\right)\beta_m^{(11)}\left(\tau\right) \quad (73b)$$

$$r_6\left(\tau\right)=\frac{1}{2}\left(\alpha_m^{(10)}\left(\tau\right)\right)^2\beta_m^{(11)}\left(\tau\right) \quad (73c)$$

for $n=m$, equation (71) becomes

$$U_{m,tt}^{(31)}+U_m^{(31)}=\frac{9}{4}\left(U_m^{(10)}\right)^2U_m^{(11)}-2\varphi_m^{-\frac{1}{2}}\mu_2'\left(\tau\right)U_{m,t\tau}^{(11)}-2\varphi_m^{-1}\mu_2'U_{m,t\tau}^{(10)}-2\varphi_m^{-\frac{1}{2}}U_{m,t\tau}^{(30)}-\varphi_m^{-1}\mu_2''U_{m,t}^{(10)}-2\varphi_m^{-\frac{1}{2}}U_{m,t}^{(30)}-2\varphi_m^{-1}\mu_2'U_{m,t}^{(10)} \quad (74)$$

Substituting for $U^{(10)}_{m,t\tau}$, $U^{(10)}_{m,t}$, $\left(U^{(10)}_{m}\right)^2 U^{(11)}_{m}$, $U^{(11)}_{m,tt}$, $U^{(30)}_{m,t\tau}$ and $U^{(30)}_{m,t\tau}$ into (74),

$$U^{(31)}_{m,tt}+U^{(31)}_{m}=\frac{9}{4}\left\{\left(r_4-\frac{1}{2}r_6\right)\sin t+\frac{1}{2}r_5\sin 2t+\frac{1}{2}r_6\sin 3t\right\}-2\varphi_m^{-\frac{1}{2}}\mu_2'\beta_m^{(11)\prime}(\tau)\cos t-$$
$$2\varphi_m^{-1}\mu_2'\alpha_m^{(10)\prime}(\tau)\sin t-2\varphi_m^{-\frac{1}{2}}\left[-\alpha_m^{(30)\prime}(\tau)\sin t+\beta_m^{(30)\prime}(\tau)\cos t+\right.$$
$$\left.\frac{3}{4}\left\{\frac{2}{3}r_2'(\tau)\sin 2t+\frac{3}{8}r_3'(\tau)\sin 3t\right\}\right]+\varphi_m^{-1}\mu_2''(\tau)\alpha_m^{(10)}(\tau)\sin t- \tag{75}$$
$$2\varphi_m^{-\frac{1}{2}}\left[-\alpha_m^{(30)}(\tau)\sin t+\beta_m^{(30)}(\tau)\cos t+\frac{3}{4}\left\{\frac{2}{3}r_2(\tau)\sin 2t+\right.\right.$$
$$\left.\left.+\frac{3}{8}r_3(\tau)\sin 3t\right\}\right]+2\varphi_m^{-1}\mu_2'(\tau)\alpha_m^{(10)}(\tau)\sin t$$

with the initial conditions

$$U^{(31)}_{m}(0,0)=0,\ U^{(31)}_{m,t}(0,0)+\varphi_m^{-\frac{1}{2}}\left\{\mu_2'(0)U^{(11)}_{m,t}(0,0)+U^{(30)}_{m,\tau}(0,0)\right\}=0 \tag{76}$$

To ensure a bounded solution in t, coefficients of $\cos t$ and $\sin t$ in (75) are equated to zero respectively and the resulting solutions are

$$\beta_m^{(30)}(\tau)=e^{-\tau}\left[\int_0^\tau G(s)e^s ds+k_1\right],k_1=\beta_m^{(30)}(0)$$
$$\beta_m^{(30)}(\tau)=e^{-\tau}\int_0^\tau G(s)e^s ds \tag{77a}$$

where

$$G(\tau)=\mu_2'(\tau)\beta_m^{(11)}(\tau) \tag{77b}$$

$$\alpha_m^{(30)}(\tau)=e^{-\tau}\left\{\int_0^\tau F(s)e^s ds+k_2\right\} \tag{77c}$$

where

$$F(\tau)=-\frac{1}{2}\varphi_m^{-\frac{1}{2}}\left[\frac{9}{4}\varphi_m^{-1}\left\{r_4(\tau)-\frac{1}{2}r_6(\tau)\right\}-2\mu_2'\varphi_m^{-1}\left\{\alpha_m^{(10)'}-\alpha_m^{(10)}\right\}+\mu_2''\varphi_m^{-1}\alpha_m^{(10)}\right] \tag{77d}$$

Solving the remaining part of equation (75), and equation (76)

$$U^{(31)}_{m}(t,\tau)=\alpha_m^{(31)}(\tau)\cos t+\beta_m^{(31)}(\tau)\sin t-\frac{1}{3}r_7(\tau)\sin 2t-\frac{1}{8}r_8(\tau)\sin 3t \tag{78}$$

$$\beta_m^{(31)}(0)=\frac{2}{3}r_7(0)+\frac{3}{8}r_8(0)-\varphi_m^{-\frac{1}{2}}\left[\mu_2'(0)\beta_m^{(11)}(0)+\alpha_m^{(30)'}(0)+\right.$$
$$\left.\frac{3}{4}\left\{r_0'(0)-\frac{1}{3}r_2'(0)-\frac{1}{8}r_3'(0)\right\}\right] \tag{79a}$$

$$\alpha_m^{(31)}(0)=0 \tag{79b}$$

where

$$r_7(\tau)=\frac{9}{8}r_5(\tau)-\varphi_m^{-\frac{1}{2}}\left\{r_2'(\tau)+r_2(\tau)\right\} \tag{80a}$$

$$r_8(\tau)=\frac{9}{8}r_6(\tau)-\frac{9}{16}\varphi_m^{-\frac{1}{2}}\left\{r_3'(\tau)+r_3(\tau)\right\} \tag{80b}$$

For $n=3m$, equation (71) becomes

$$U^{(31)}_{3m,tt}+\Omega_m^2U^{(31)}_{3m}=-\frac{3}{4}\left\{\left(U^{(10)}_{m}\right)^2U^{(11)}_{m}\right\}-2\varphi_m^{-\frac{1}{2}}U^{(30)}_{3m,t\tau}-2\varphi_m^{-\frac{1}{2}}U^{(30)}_{3m,t} \tag{81}$$

with initial conditions

$$U_{3m}^{(31)}(0,0)=0,\ U_{3m,t}^{(31)}(0,0)+\varphi_m^{-\frac{1}{2}}U_{3m,\tau}^{(30)}(0,0)=0 \tag{82}$$

Substituting for $\left(U_m^{(10)}\right)^2 U_m^{(11)}$, $U_{m,t\tau}^{(30)}$ and $U_{m,t}^{(30)}$ into (81),

$$\begin{aligned}U_{3m,tt}^{(31)}+\Omega_m^2U_{3m}^{(31)}=&-\frac{3}{4}\left\{\left(r_4-\frac{1}{2}r_6\right)\sin t+\frac{1}{2}r_5\sin 2t+\frac{1}{2}r_6\sin 3t\right\}-\\ &2\varphi_m^{-\frac{1}{2}}\left\{-\Omega_m\alpha_{3m}^{(30)\prime}(\tau)\sin\Omega_m t+\Omega_m\beta_{3m}^{(30)\prime}(\tau)\cos\Omega_m t\right.\\ &\left.+\frac{1}{4}\left(\frac{r_1'(\tau)\sin t}{\Omega_m^2-1}+\frac{2r_2'(\tau)\sin 2t}{\Omega_m^2-4}+\frac{3r_3'(\tau)\sin 3t}{\Omega_m^2-9}\right)\right\}-\\ &2\varphi_m^{-\frac{1}{2}}\left\{-\Omega_m\alpha_{3m}^{(30)}(\tau)\sin\Omega_m t+\Omega_m\beta_{3m}^{(30)}(\tau)\cos\Omega_m t+\right.\\ &\left.\frac{1}{4}\left(\frac{r_1(\tau)\sin t}{\Omega_m^2-1}+\frac{2r_2(\tau)\sin 2t}{\Omega_m^2-4}+\frac{3r_3(\tau)\sin 3t}{\Omega_m^2-9}\right)\right\}\end{aligned} \tag{83}$$

To ensure a bounded solution in t, coefficients of $\cos\Omega_m t$ and $\sin\Omega_m t$ in (83) are equated to zero respectively and the resulting solutions are

$$\beta_{3m}^{(30)}(\tau)=0 \tag{84a}$$

$$\alpha_{3m}^{(30)}(\tau)=\frac{1}{4}\varphi_m^{-1}e^{-\tau}\left(B_m\bar{a}_m\right)^3\left\{\frac{5}{2\Omega_m^2}-\frac{15}{4\left(\Omega_m^2-1\right)}+\frac{3}{2\left(\Omega_m^2-4\right)}-\frac{1}{4\left(\Omega_m^2-9\right)}\right\} \tag{84b}$$

The remaining part of (83) is

$$U_{3m,tt}^{(31)}+\Omega_m^2U_{3m}^{(31)}=r_9(\tau)\sin t+r_{10}(\tau)\sin 2t+r_{11}(\tau)\sin 3t \tag{85}$$

where

$$r_9(\tau)=-\frac{3}{4}\varphi_m^{-1}\left\{r_4(\tau)-\frac{1}{2}r_6(\tau)\right\}-\frac{1}{2\varphi_m^{\frac{1}{2}}\left(\Omega_m^2-1\right)}\left\{r_1'(\tau)+r_1(\tau)\right\} \tag{86}$$

$$r_{10}(\tau)=-\frac{3}{8}\varphi_m^{-1}r_5(\tau)-\frac{1}{\varphi_m^{\frac{1}{2}}\left(\Omega_m^2-4\right)}\left\{r_2'(\tau)+r_2(\tau)\right\} \tag{87}$$

$$r_{11}(\tau)=-\frac{3}{8}\varphi_m^{-1}r_6(\tau)-\frac{3}{2\varphi_m^{\frac{1}{2}}\left(\Omega_m^2-9\right)}\left\{r_3'(\tau)+r_3(\tau)\right\} \tag{88}$$

Solving the remaining part of equation (85), and equation (82)

$$\begin{aligned}U_{3m}^{(31)}(t,\tau)=&\alpha_{3m}^{(31)}(\tau)\cos\Omega_m t+\beta_{3m}^{(31)}(\tau)\sin\Omega_m t+\frac{1}{\Omega_m^2-1}r_9(\tau)\sin t+\\ &\frac{1}{\Omega_m^2-4}r_{10}(\tau)\sin 2t+\frac{1}{\Omega_m^2-9}r_{11}(\tau)\sin 3t\end{aligned} \tag{89}$$

$$\begin{aligned}\beta_{3m}^{(31)}(0)=\frac{1}{\Omega_m}\Bigg[&-\left(\frac{r_9(0)}{\Omega_m^2-1}+\frac{2r_{10}(0)}{\Omega_m^2-4}+\frac{3r_{11}(0)}{\Omega_m^2-9}\right)-\\ &\varphi_m^{-\frac{1}{2}}\left\{\alpha_{3m,\tau}^{(30)}(0)-\frac{1}{4}\left(\frac{r_0'(0)}{\Omega_m^2}+\frac{r_1'(0)}{\Omega_m^2-1}+\frac{r_2'(0)}{\Omega_m^2-4}+\frac{r_3'(0)}{\Omega_m^2-9}\right)\right\}\Bigg]\end{aligned} \tag{90a}$$

$$\alpha_{3m}^{(31)}(0)=0 \quad (90b)$$

From equation (18)

$$U^{(31)}=U_m^{(31)}\sin mx+U_{3m}^{(31)}\sin 3mx \quad (91)$$

Solutions of equation $\varepsilon^3.\delta^2$

$$\begin{aligned}\sum_{n=1}^{\infty}\left[U_{n,tt}^{(32)}+\frac{\varphi_n}{\varphi_m}U_n^{(32)}\right]\sin nx=3\eta\varphi_m^{-1}\left[\frac{1}{4}\left\{\left(U_m^{(10)}\right)^2\left(U_m^{(12)}\right)\right\}(3\sin mx-\sin 3mx)+\right.\\ \left.\frac{1}{4}\left\{\left(U_m^{(10)}\right)\left(U_m^{(11)}\right)^2\right\}(3\sin mx-\sin 3mx)\right]-\\ 2\varphi_m^{-\frac{1}{2}}\mu_2'\left(U_{m,tt}^{(12)}\sin t\right)-2\varphi_m^{-1}\mu_2'\left(U_{m,t\tau}^{(11)}\sin mx\right)-\\ 2\varphi_m^{-\frac{1}{2}}\left(U_{m,tt}^{(31)}\sin mx+U_{3m,tt}^{(31)}\sin 3mx\right)-\\ \varphi_m^{-1}\mu_2''\left(U_{m,t}^{(11)}\sin mx\right)-\\ 2\varphi_m^{-\frac{1}{2}}\left(U_{m,t}^{(31)}\sin mx+U_{3m,t}^{(31)}\sin 3mx\right)-\\ 2\varphi_m^{-1}\mu_2'\left(U_{m,t}^{(11)}\sin mx\right)-\varphi_m^{-1}\left(U_{m,\tau\tau}^{(30)}\sin mx+\right.\\ \left.U_{3m,\tau\tau}^{(30)}\sin 3mx\right)-2\varphi_m^{-1}\left(U_{m,\tau}^{(30)}\sin mx+U_{3m,\tau}^{(30)}\sin 3mx\right)\end{aligned} \quad (92)$$

for $n=m$, equation (92) becomes

$$\begin{aligned}U_{m,tt}^{(32)}+U_m^{(32)}=\frac{9}{4}\left(U_m^{(10)}\right)^2U_m^{(12)}+\frac{3}{4}U_m^{(10)}\left(U_m^{(11)}\right)^2-2\varphi_m^{-\frac{1}{2}}\mu_2'U_{m,tt}^{(12)}-\\ 2\varphi_m^{-1}\mu_2'U_{m,t\tau}^{(11)}-2\varphi_m^{-\frac{1}{2}}U_{m,t\tau}^{(31)}-\varphi_m^{-1}\mu_2''U_{m,t}^{(11)}-2\varphi_m^{-\frac{1}{2}}U_{m,t}^{(31)}-\\ 2\varphi_m^{-1}\mu_2'U_{m,t}^{(11)}-\varphi_m^{-1}U_{m,\tau\tau}^{(30)}-2\varphi_m^{-1}U_{m,\tau}^{(30)}\end{aligned} \quad (93)$$

with initial conditions

$$U_m^{(32)}(0,0)=0,\ U_{m,t}^{(32)}(0,0)+\varphi_m^{-\frac{1}{2}}\left(\mu_2'(0)U_{m,t}^{(12)}(0,0)+U_{m,\tau}^{(31)}(0,0)\right)=0 \quad (94)$$

Expanding the terms, $\frac{9}{4}\left(U_m^{(10)}\right)^2U_m^{(12)}$ and $\frac{3}{4}U_m^{(10)}\left(U_m^{(11)}\right)^2$ in equation (93), it gives

$$\begin{aligned}\frac{9}{4}\left(U_m^{(10)}\right)^2U_m^{(12)}=\frac{9}{4}\left\{\alpha_m^{12}(\tau)r_{12}(\tau)\cos t+\frac{1}{2}\alpha_m^{(12)}(\tau)r_{13}(\tau)(1+\cos 2t)+\right.\\ \frac{1}{2}\alpha_m^{(12)}(\tau)r_{14}(\tau)(\cos 3t+\cos t)+\beta_m^{(12)}(\tau)r_{12}(\tau)\sin t+\\ \left.\frac{1}{2}\beta_m^{(12)}(\tau)r_{13}(\tau)\sin 2t+\frac{1}{2}\beta_m^{(12)}(\tau)r_{14}(\sin 3t-\sin t)\right\}\end{aligned} \quad (95)$$

where

$$r_{12}(\tau)=\frac{1}{2}\left(\alpha_m^{(10)}(\tau)\right)^2+\left(B_m\bar{a}_m\right)^2 \quad (96)$$

$$r_{13}(\tau)=2B_m\bar{a}_m\alpha_m^{(10)}(\tau) \quad (97)$$

$$r_{14}(\tau)=\frac{1}{2}\left(\alpha_m^{(10)}(\tau)\right)^2 \quad (98)$$

$$\frac{3}{4}U_m^{(10)}\left(U_m^{(11)}\right)^2=\frac{3}{8}\left(\beta_m^{(11)}\right)^2\left\{B_m\bar{a}_m+\frac{1}{2}\alpha_m^{(10)}(\tau)\cos t-B_m\bar{a}_m\cos 2t-\frac{1}{2}\alpha_m^{(10)}(\tau)\cos 3t\right\} \quad (99)$$

Then, Substituting for the following terms;

$\frac{9}{4}\left(U_m^{(10)}\right)^2U_m^{(12)},\ \frac{3}{4}U_m^{(10)}\left(U_m^{(11)}\right)^2,\ U_{m,t\tau}^{(11)},\ U_{m,t}^{(11)},\ U_{m,\tau\tau}^{(30)},\ U_{m,\tau}^{(30)},\ U_{m,t\tau}^{(31)},\ and\ U_{m,\tau}^{(31)}$ in equation (93), equation (93) becomes

$$\begin{aligned}U_{m,tt}^{(32)}+U_m^{(32)}=&\frac{9}{4}\Bigg[\frac{1}{2}\alpha_m^{(12)}(\tau)r_{13}(\tau)+\left\{\alpha_m^{(12)}(\tau)r_{12}(\tau)+\frac{1}{2}\alpha_m^{(12)}(\tau)r_{14}(\tau)\right\}\cos t+\\&\left\{\beta_m^{(12)}(\tau)r_{12}(\tau)-\frac{1}{2}\beta_m^{(12)}(\tau)r_{14}(\tau)\right\}\sin t+\frac{1}{2}\alpha_m^{(12)}(\tau)r_{13}(\tau)\cos 2t+\\&\frac{1}{2}\beta_m^{(12)}(\tau)r_{13}(\tau)\sin 2t+\frac{1}{2}r_{14}(\tau)\left\{\alpha_m^{(12)}(\tau)\cos 3t+\beta_m^{(12)}(\tau)\sin 3t\right\}\Bigg]+\\&\frac{3}{8}\left(\beta_m^{(11)}\right)^2\left\{B_m\bar{a}_m+\frac{1}{2}\alpha_m^{(10)}(\tau)\cos t-B_m\bar{a}_m\cos 2t-\frac{1}{2}\left(\alpha_m^{(10)}(\tau)\right)\cos 3t\right\}+\\&2\varphi_m^{-\frac{1}{2}}\mu_2'(\tau)\left\{\alpha_m^{(12)}(\tau)\cos t+\beta_m^{(12)}(\tau)\sin t\right\}-2\varphi_m^{-1}\mu_2'(\tau)\Big\{-\alpha_m^{(12)\prime}(\tau)\sin t+\\&\beta_m^{(12)\prime}(\tau)\cos t\Big\}-\varphi_m^{-1}\mu_2''(\tau)\beta_m^{(11)}(\tau)\cos t-2\varphi_m^{-1}\mu_2'(\tau)\beta_m^{(11)}(\tau)\cos t-\\&\varphi_m^{-\frac{1}{2}}\left\{-\alpha_m^{(31)\prime}(\tau)\sin t+\beta_m^{(31)\prime}(\tau)\cos t-\frac{2}{3}r_7'(\tau)\cos 2t-\frac{3}{8}r_8'(\tau)\cos 3t\right\}-\\&2\varphi_m^{-\frac{1}{2}}\left\{-\alpha_m^{(31)}(\tau)\sin t+\beta_m^{(31)}(\tau)\cos t-\frac{2}{3}r_7(\tau)\cos 2t-\frac{3}{8}r_8(\tau)\cos 3t\right\}-\\&\varphi_m^{-1}\Bigg\{\alpha_m^{(30)\prime\prime}(\tau)\cos t+\beta_m^{(30)\prime\prime}(\tau)\sin t+\frac{3}{4}\Bigg(r_0''(\tau)-\frac{1}{3}r_2''(\tau)\cos 2t-\\&\frac{1}{8}r_3''(\tau)\cos 3t\Bigg)\Bigg\}-2\varphi_m^{-1}\Bigg\{\alpha_m^{(30)\prime}(\tau)\cos t+\beta_m^{(30)\prime}(\tau)\sin t+\frac{3}{4}\Bigg(r_0'(\tau)-\\&\frac{1}{3}r_2'(\tau)\cos 2t-\frac{1}{8}r_3'(\tau)\cos 3t\Bigg)\Bigg\}\end{aligned} \quad (100)$$

To ensure a bounded solution in t, coefficients of $\cos t$ and $\sin t$ in (30) are equated to zero respectively and the resulting solutions are

$$\beta_m^{(31)}(\tau)=e^{-\tau}\left\{\int_0^\tau H(s)e^s ds+k_3\right\} \quad (101)$$

where

$$k_3=\beta_m^{(31)}(0)\neq 0 \quad (102a)$$

$$\begin{aligned}H(\tau)=&-\frac{1}{2}\varphi_m^{-\frac{1}{2}}\Bigg[\frac{9}{4}\varphi_m^{-1}\alpha_m^{(12)}(\tau)\left(r_{12}(\tau)+r_{14}(\tau)\right)+\frac{3}{16}\varphi_m^{-1}\left(\beta_m^{(11)}(\tau)\right)^2\alpha_m^{(10)}(\tau)-\\&2\varphi_m^{-\frac{1}{2}}\mu'(\tau)_2\,\alpha_m^{(12)}(\tau)-2\varphi_m^{-1}\mu_2'(\tau)\beta_m^{(12)\prime}(\tau)-\varphi_m^{-1}\mu_2''(\tau)\beta_m^{(11)}(\tau)-\\&2\varphi_m^{-1}\mu_2'(\tau)\beta_m^{(11)}(\tau)-\varphi_m^{-1}\alpha_m^{(30)\prime\prime}(\tau)-2\varphi_m^{-1}\alpha_m^{(30)\prime}(\tau)\Bigg]\end{aligned} \quad (102b)$$

$$\alpha_m^{(31)}(\tau)=e^{-\tau}\left\{\int_0^\tau P(s)e^s ds+k_4\right\} \quad (103)$$

where

$$k_4 = \alpha_m^{(31)}(0) = 0 \tag{104a}$$

$$P(\tau) = \frac{1}{2}\varphi_m^{-\frac{1}{2}}\left[\frac{9}{4}\varphi_m^{-1}\beta_m^{(12)}(\tau)\left\{r_{12}(\tau) - \frac{1}{2}r_{14}(\tau)\right\} + 2\varphi_m^{-\frac{1}{2}}\mu_2'(\tau)\beta_m^{(12)}(\tau) + 2\varphi_m^{-1}\mu_2'(\tau)\alpha_m^{(12)\prime}(\tau) - \varphi_m^{-1}\beta_m^{(30)\prime\prime}(\tau) - 2\varphi_m^{-1}\beta_m^{(30)}(\tau)\right] \tag{104b}$$

The remaining part of equation (100) is

$$U_{m,tt}^{(32)} + U_m^{(32)} = r_{15}(\tau) + r_{16}(\tau)\cos 2t + r_7(\tau)\sin t + r_{18}(\tau)\cos 3t + r_{19}(\tau)\sin 3t \tag{105}$$

where

$$r_{15}(\tau) = \frac{3}{8}\varphi_m^{-1}\left\{r_{13}(\tau)\alpha_m^{(12)}(\tau) + B_m\bar{a}_m\left(\beta_m^{(11)}(\tau)\right)^2\right\} - \frac{3}{4}\varphi_m^{-1}\left\{r_0''(\tau) + \frac{1}{2}r_0'(\tau)\right\} \tag{106}$$

$$r_{16}(\tau) = \frac{3}{8}\varphi_m^{-1}\left\{3r_{13}(\tau)\alpha_m^{(12)}(\tau) - B_m\bar{a}_m\left(\beta_m^{(11)}(\tau)\right)^2\right\} + \frac{4}{3}\varphi_m^{-\frac{1}{2}}\left\{r_7'(\tau) + r_7(\tau)\right\} + \frac{1}{3}\varphi_m^{-1}\left\{r_2''(\tau) + r_2'(\tau)\right\}$$

(107) $r_{17}(\tau) = \frac{9}{8}\varphi_m^{-1}r_{13}(\tau)\beta_m^{(12)}(\tau)$ (108)

$$r_{18}(\tau) = \frac{3}{8}\varphi_m^{-1}\left\{3r_{14}(\tau)\alpha_m^{(12)}(\tau) - \frac{1}{2}\alpha_m^{(10)}(\tau)\left(\beta_m^{(11)}(\tau)\right)^2\right\} + \frac{6}{8}\varphi_m^{-\frac{1}{2}}\left\{r_8'(\tau) + r_8(\tau)\right\} + \frac{1}{8}\varphi_m^{-1}\left\{r_3''(\tau) + 2r_3'(\tau)\right\}$$

(109) Solving (105), and (94),

$$U_m^{(32)}(t,\tau) = \alpha_m^{(32)}(\tau)\cos t + \beta_m^{(32)}(\tau)\sin t + r_{15}(\tau) - \frac{1}{3}r_{16}(\tau)\cos 2t - \frac{1}{3}r_{17}(\tau)\sin t - \frac{1}{8}r_{18}(\tau)\cos 3t - \frac{1}{8}r_{19}(\tau)\sin 3t \tag{110}$$

$$\beta_m^{(32)}(0) = \frac{2}{3}r_{17}(0) - \frac{3}{8}r_{19}(0) \tag{111}$$

$$\alpha_m^{(32)}(0) = -r_{15}(0) - \frac{1}{3}r_{16}(0) + \frac{1}{8}r_{18}(0) \tag{112}$$

for $n = 3m$, equation (92) becomes

$$U_{3m,tt}^{(32)} + \Omega_m^2 U_{3m}^{(32)} = -\frac{3}{4}\left\{\left(U_m^{(10)}\right)^2 U_m^{(12)} + U_m^{(10)}\left(U_m^{(11)}\right)^2\right\} - 2\varphi_m^{-\frac{1}{2}}U_{3m,t\tau}^{(31)} - 2\varphi_m^{-\frac{1}{2}}U_{3m,t}^{(31)} - \varphi_m^{-1}U_{3m,\tau\tau}^{(30)} - 2\varphi_m^{-1}U_{3m,\tau}^{(30)} \tag{113}$$

with initial conditions

$$U_{3m}^{(32)}(0,0) = 0,\ U_{3m,t}^{(32)}(0,0) + \varphi_m^{-\frac{1}{2}}\left(U_{3m,\tau}^{(31)}(0,0)\right) = 0 \tag{114}$$

Further expansion of equation (113) yeilds

$$U_{3m,tt}^{(32)}+\Omega_m^2U_{3m}^{(32)}=-\frac{3}{4}\left\{\alpha_m^{(12)}(t,\tau)\left(\frac{1}{2}r_{13}(\tau)+r_{12}(\tau)+r_{14}(\tau)\right)\cos t+\right.$$
$$\beta_m^{(12)}(t,\tau)\left(r_{12}(\tau)-\frac{1}{2}r_{14}(\tau)\right)\sin t+\frac{1}{2}\alpha_m^{(12)}(t,\tau)r_{13}\cos t+$$
$$\left.\frac{1}{2}\beta_m^{(12)}(t,\tau)r_{14}(\tau)\sin 2t+\frac{1}{2}r_{14}(\tau)\left(\alpha_m^{(12)}(t,\tau)\cos 3t+\beta_m^{(12)}(t,\tau)\sin 3t\right)\right\}-$$
$$\frac{3}{8}\left(\beta_m^{(11)}(t,\tau)\right)^2\left\{B_m\bar{a}_m(1-\cos 2t)+\frac{1}{2}\alpha_m^{(10)}(t,\tau)\left(\cos t-\alpha_m^{(10)}(t,\tau)\cos 3t\right)\right\}-$$
$$2\varphi_m^{-\frac{1}{2}}\left\{-\Omega_m\alpha_{3m}^{(31)\prime}(t,\tau)\sin\Omega_m t+\Omega_m\beta_{3m}^{(31)\prime}(t,\tau)\cos\Omega_m t+\frac{r_9'\cos t}{\Omega_m^2-1}+\right.$$
$$\left.\frac{2r_{10}'(\tau)\cos 2t}{\Omega_m^2-4}+\frac{3r_{11}'(\tau)\cos 3t}{\Omega_m^2-9}\right\}-2\varphi_m^{-\frac{1}{2}}\left\{-\Omega_m\alpha_{3m}^{(31)}(t,\tau)\sin\Omega_m t+\right.$$
$$\left.\Omega_m\beta_{3m}^{(31)}(t,\tau)\cos\Omega_m t+\frac{r_9(\tau)\cos t}{\Omega_m^2-1}+\frac{2r_{10}(\tau)\cos 2t}{\Omega_m^2-4}+\frac{3r_{11}(\tau)\cos 3t}{\Omega_m^2-9}\right\}+$$
$$\varphi_m^{-1}\left\{\alpha_{3m}^{(30)\prime\prime}(t,\tau)\cos\Omega_m t+\beta_{3m}^{(30)\prime\prime}(t,\tau)\sin\Omega_m t-\frac{1}{4}\left(\frac{r_0''(\tau)}{\Omega_m^2}+\frac{r_1''(\tau)\cos t}{\Omega_m^2-1}+\right.\right.$$
$$\left.\left.\frac{r_2''(\tau)\cos 2t}{\Omega_m^2-4}+\frac{r_3''(\tau)\cos 3t}{\Omega_m^2-9}\right)\right\}-2\varphi_m^{-1}\left\{\alpha_{3m}^{(30)\prime}(t,\tau)\cos\Omega_m t+\right.$$
$$\left.\beta_{3m}^{(30)\prime}(t,\tau)\sin\Omega_m t-\frac{1}{4}\left(\frac{r_0'(\tau)}{\Omega_m^2}+\frac{r_1'(\tau)\cos t}{\Omega_m^2-1}+\frac{r_2(\tau)\cos 2t}{\Omega_m^2-4}+\frac{r_3(\tau)\cos 3t}{\Omega_m^2-9}\right)\right\}$$
(115)

To ensure a bounded solution in t, coefficients of $\cos\Omega_m t$ and $\sin\Omega_m t$ in (83) are equated to zero respectively and the resulting solutions are

$$\beta_{3m}^{(31)}(\tau)=e^{-\tau}\int_0^\tau I(s)e^s ds+k_5 \quad (116)$$

where

$$k_5=\beta_{3m}^{(30)}(0)\neq 0 \quad (117a)$$

$$I(\tau)=-\frac{1}{2}\varphi_m^{-\frac{1}{2}}\Omega_m^{-1}\left\{\alpha_{3m}^{(30)\prime\prime}(\tau)+2\alpha_{3m}^{(30)\prime}(\tau)\right\} \quad (117b)\ \alpha_{3m}^{(31)}(\tau)=0$$

(118)

The remaining part of equation (115) is

$$U_{3m,tt}^{(32)}+\Omega_m^2U_{3m}^{(32)}=r_{20}(\tau)+r_{21}(\tau)\sin t+r_{22}(\tau)\cos t+r_{23}(\tau)\sin 2t+$$
$$r_{24}(\tau)\cos 2t+r_{25}(\tau)\sin 3t+r_{26}(\tau)\cos 3t \quad (119)$$

where

$$r_{20}(\tau)=\frac{1}{2\Omega_m^2}\varphi_m^{-1}\left\{\frac{1}{2}r_0''(\tau)+r_0'(\tau)\right\} \quad (120)$$

$$r_{21}(\tau) = -\frac{3}{4}\beta_m^{(12)}(\tau)\left\{r_{12}(\tau) - \frac{1}{2}r_{14}(\tau)\right\} \quad (121)$$

$$r_{22}(\tau) = -\frac{3}{8}\alpha_m^{(12)}(\tau)\{2r_{12}(\tau) + r_{13}(\tau) + r_{14}(\tau)\} - \frac{3}{16}\alpha_m^{(10)}(\tau)\left(\beta_m^{(11)}(\tau)\right)^2 - \frac{2}{\Omega_m^2 - 1}\varphi_m^{-\frac{1}{2}}\{r_9'(\tau) + r_9(\tau)\} + \frac{1}{4(\Omega_m^2 - 1)}\varphi_m^{-1}\{r_1''(\tau) + 2r_1'(\tau)\} \quad (122)$$

$$r_{23}(\tau) = -\frac{3}{8}r_4(\tau)\beta_m^{(12)}(\tau) \quad (123)$$

$$r_{24}(\tau) = -\frac{3}{8}\left\{\alpha_m^{(12)}(\tau)r_{13}(\tau) - B_m\bar{a}_m\left(\beta_m^{(11)}(\tau)\right)^2\right\} - \frac{4r_0'}{(\Omega_m^2 - 1)\varphi_m^{\frac{1}{2}}} - \frac{1}{4(\Omega_m^2 - 4)}\left\{16\varphi_m^{-\frac{1}{2}}r_{10}(\tau) - \varphi_m^{-1}r_2''(\tau) + 2\varphi_m^{-1}r_2(\tau)\right\} \quad (124)$$

$$r_{25}(\tau) = \frac{3}{8}r_{14}(\tau)\beta_m^{(12)}(\tau) \quad (125)$$

$$r_{26}(\tau) = -\frac{3}{8}\alpha_m^{(12)}(\tau)r_{14}(\tau) + \frac{3}{16}\left(\alpha_m^{(10)}(\tau)\right)^2\left(\beta_m^{(11)}(\tau)\right)^2 - \frac{6}{(\Omega_m^2 - 9)\varphi_m^{\frac{1}{2}}}\{r_{11}'(\tau) + r_{11}(\tau)\} + \frac{1}{4(\Omega_m^2 - 9)\varphi_m^{1}}\{r_3''(\tau) + 2r_3(\tau)\} \quad (126)$$

The solutions of equation (115) and (114) are

$$U_{3m}^{(32)}(\tau) = \alpha_{3m}^{(32)}(\tau)\cos\Omega_m t + \beta_{3m}^{(32)}(\tau)\sin\Omega_m t + \frac{r_{21}(\tau)}{\Omega_m^2 - 1}\sin t + \frac{r_{23}(\tau)\sin 2t}{\Omega_m^2 - 4} + \frac{r_{25}(\tau)\sin 3t}{\Omega_m^2 - 9} + \frac{r_{22}(\tau)\cos t}{\Omega_m^2 - 1} + \frac{r_{24}(\tau)\cos 2t}{\Omega_m^2 - 4} + \frac{r_{26}(\tau)\cos 3t}{\Omega_m^2 - 9} \quad (127)$$

$$\beta_{3m}^{(32)}(0) = -\frac{1}{\Omega_m}\left(\frac{r_{21}(0)}{\Omega_m^2 - 1} + \frac{2r_{23}(0)}{\Omega_m^2 - 4} + \frac{3r_{25}(0)}{\Omega_m^2 - 9}\right) \quad (128)$$

$$\alpha_{3m}^{(32)}(0) = -\left(\frac{r_{22}(0)}{\Omega_m^2 - 1} + \frac{r_{24}(0)}{\Omega_m^2 - 4} + \frac{r_{26}(0)}{\Omega_m^2 - 9}\right) \quad (129)$$

The displacement $U(x,t,\tau)$ from equation (7) becomes

$$U(x,t,\tau) = \varepsilon\left(U_m^{(10)}\sin mx + \delta U_m^{(11)}\sin mx + \delta^2 U_m^{(12)}\sin mx\right) + \varepsilon^3\left\{\left(U_m^{(30)}\sin mx + U_{3m}^{(30)}\sin 3mx\right) + \delta\left(U_m^{(31)}\sin mx + U_{3m}^{(31)}\sin 3mx\right) + \delta^2\left(U_m^{(32)}\sin mx + U_{3m}^{(32)}\sin 3mx\right)\right\} + \ldots \quad (130)$$

7. MAXIMUM DISPLACEMENT

Let $U_a(x_a,t_a,\tau_a)$ be the maximum displacement, where x_a is the space variable and t_a,τ_a are the time variables at maximum displacement. For simplification, the following asymptotic expansions shall be assumed as in Onuoha and Ette [17]

$$\bar{t}_a = \sum_{i=0}^{\infty} \varepsilon^i \left(\sum_{j=0}^{\infty} \bar{t}_{ij} \delta^j \right) \quad (131) \qquad t_a = \sum_{i=0}^{\infty} \varepsilon^i \left(\sum_{j=0}^{\infty} t_{ij} \delta^j \right) \quad (132)$$

$$\tau_a = \delta \left\{ \sum_{i=0}^{\infty} \varepsilon^i \left(\sum_{j=0}^{\infty} \bar{t}_{ij} \delta^j \right) \right\} \quad (133)$$

The conditions for maximum displacement U_a are

$$\frac{\partial U}{\partial x} = 0 \quad (134a)$$

and

$$\left\{ \left(m^4 - 2m^2\lambda + 1 \right)^{\frac{1}{2}} + \mu_2'(\tau)\varepsilon^2 + \ldots \right\} U_{,t} + \delta U_{,\tau} = 0 \quad (134b)$$

Equation (134a) from equation (130) gives

$$\begin{aligned} &\varepsilon\left[mU_m^{(10)}(t,\tau)\cos mx + \delta m U_m^{(10)}(t,\tau)\cos mx + m\delta^2 U_m^{(12)}(t,\tau)\cos mx + \ldots \right] + \\ &\varepsilon^3\Big[\left(mU_m^{(30)}(t,\tau)\cos mx + 3mU_{3m}^{(30)}(t,\tau)\cos 3mx \right) + \\ &\delta\left(mU_m^{(31)}(t,\tau)\cos mx + 3mU_{3m}^{(31)}(t,\tau)\cos 3mx \right) + \\ &\delta^2\left(mU_m^{(32)}(t,\tau)\cos mx + 3mU_{3m}^{(32)}(t,\tau)\cos 3mx \right)\Big] + \ldots = 0 \end{aligned} \quad (135)$$

At maximum displacement, $x = x_a$, and from equation (135),

$$\cos m x_a = 0 \quad (136a)$$

Hence,

$$x_a = \frac{\pi}{2m} \quad (136b)$$

Equation (134b) for $x = x_a$ becomes

$$\begin{aligned} &\left[1 + \varphi_m^{-\frac{1}{2}}\left(\mu_2'(\tau)\varepsilon^2 + \mu_3'(\tau)\varepsilon^3 + \ldots \right) \right]\Big[\varepsilon\left\{ U_{m,t}^{(10)}(t,\tau) + \delta U_{m,t}^{(11)}(t,\tau) + \right. \\ &\delta^2 U_{m,t}^{(12)}(t,\tau) + \varepsilon^3\left\{ U_{m,t}^{(30)}(t,\tau) - U_{3m,t}^{(30)}(t,\tau) + \delta\left(U_{m,t}^{(31)}(t,\tau) - U_{3m,t}^{(31)}(t,\tau) \right) + \right. \\ &\left.\left.\delta^2\left(U_{m,t}^{(32)}(t,\tau) - U_{3m,t}^{(32)}(t,\tau) \right) \right\}\Big] + \delta\varphi_m^{-\frac{1}{2}}\Big[\varepsilon\left\{ U_{m,\tau}^{(10)}(t,\tau) + \delta U_{m,\tau}^{(11)}(t,\tau) + \right. \\ &\delta^2 U_{m,\tau}^{(12)}(t,\tau) + \varepsilon^3\left\{ U_{m,\tau}^{(30)}(t,\tau) - U_{3m,\tau}^{(30)}(t,\tau) + \delta\left(U_{m,\tau}^{(31)}(t,\tau) - U_{3m,\tau}^{(31)}(t,\tau) \right) + \right. \\ &\left.\left.\delta^2\left(U_{m,\tau}^{(32)} - U_{3m,\tau}^{(32)} \right) \right\}\Big] + \ldots = 0 \end{aligned} \quad (137)$$

At maximum displacement,

$$U_a = \varepsilon\Big[U_m^{(10)}(t_0,0)+\delta\left\{t_{01}U_{m,t}^{(10)}+\bar{t}_0U_{m,\tau}^{(10)}+U_m^{(11)}\right\}+\delta^2\left\{t_{02}U_{m,t}^{(10)}+\bar{t}_0U_{m,\tau}^{(10)}+t_{01}^2U_{m,tt}^{(10)}+\right.$$

$$\left.2\bar{t}_0t_{01}U_{m,t\tau}^{(10)}+\frac{1}{2}\bar{t}_0^{\,2}U_{m,\tau\tau}^{(10)}+t_{01}U_{m,t}^{(11)}+\bar{t}_0U_{m,\tau}^{(11)}+U_m^{(12)}\Big]+\right.$$

$$\varepsilon^3\Big[\left\{t_{20}U_{m,t}^{(10)}+\frac{1}{2}t_{10}U_{m,tt}^{(10)}+\frac{1}{2}t_{10}U_{m,t\tau}^{(10)}+U_m^{(30)}-U_{3m}^{(30)}\right\}+\delta\left\{t_{21}U_{m,t}^{(10)}+\bar{t}_{20}U_{m,\tau}^{(10)}+\frac{1}{2}t_{10}t_{11}U_{m,tt}^{(10)}+\right.$$

$$\left.t_{10}\bar{t}_{10}U_{m,t\tau}^{(10)}+t_{20}\bar{t}_0U_{m,t\tau}^{(10)}+t_{20}U_{m,t}^{(11)}+\frac{1}{2}t_{10}U_{m,tt}^{(11)}+t_{01}U_{m,t}^{(30)}+\bar{t}_0\left(U_{m,\tau}^{(30)}+U_m^{(31)}\right)-\left(t_{01}U_{3m,t}^{(30)}+\bar{t}_0U_{3m,\tau}^{(30)}+U_{3m}^{(31)}\right)\right\}+$$

$$\delta^2\left\{\left(t_{22}U_{m,t}^{(10)}+\bar{t}_{21}U_{m,\tau}^{(10)}+\frac{1}{2}t_{11}^2U_{m,tt}^{(10)}+\left(t_{20}\bar{t}_{01}+t_{21}\bar{t}_0+t_{11}\bar{t}_{10}+t_{10}\bar{t}_{11}+t_{01}\bar{t}_{20}\right)U_{m,t\tau}^{(10)}+\bar{t}_{10}^{\,2}U_{m,\tau\tau}^{(10)}+t_{21}U_{m,t}^{(11)}+\right.\right.$$

$$\bar{t}_{20}U_{m,\tau}^{(11)}+\frac{1}{2}\left(t_{11}^2+2t_{10}t_{12}\right)U_{m,tt}^{(11)}+\left(t_{10}\bar{t}_{10}+t_{20}\bar{t}_0\right)U_{m,t\tau}^{(11)}+t_{20}U_{m,t}^{(12)}+\frac{1}{2}t_{10}^2U_{m,tt}^{(12)}+t_{02}U_{m,t}^{(30)}+\bar{t}_{01}U_{m,\tau}^{(30)}+$$

$$\left.\frac{1}{2}t_{01}^2U_{m,tt}^{(30)}+\bar{t}_0t_{01}U_{m,t\tau}^{(30)}+\bar{t}_0^{\,2}U_{m,\tau\tau}^{(30)}+t_{01}U_{m,t}^{(31)}+\bar{t}_0U_{m,\tau}^{(31)}+\frac{1}{2}t_{01}U_{m,tt}^{(31)}+U_m^{(32)}\right)-$$

$$\left.\left.\left(t_{02}U_{3m,t}^{(30)}+\bar{t}_{01}U_{3m,\tau}^{(30)}+\frac{1}{2}t_{01}U_{3m,tt}^{(30)}+\bar{t}_0t_{01}U_{3m,t\tau}^{(30)}+\bar{t}_0^{\,2}U_{3m,\tau\tau}^{(30)}+t_{01}U_{3m,t}^{(31)}+\bar{t}_0U_{3m,\tau}^{(31)}+\frac{1}{2}t_{01}U_{3m}^{(31)}+U_{3m}^{(32)}\right)\right\}\right]$$

(138)

Equate to zero in (138) the coefficients of powers of $\varepsilon^i\delta^j$, $i=1,2,\ldots;\ j=0,1,\ldots$ and get

$$(\varepsilon.1): U_{m,t}^{(10)}(t_0,0)=0 \tag{139}$$

$$(\varepsilon.\delta): U_{m,tt}^{(10)}t_{01}+\bar{t}_0U_{m,t\tau}^{(10)}+U_{m,t}^{(11)}+\varphi_m^{-\frac{1}{2}}U_{m,\tau}^{(10)}=0 \tag{140}$$

$$(\varepsilon.\delta^2): t_{02}U_{m,tt}^{(10)}+\bar{t}_{01}U_{m,t\tau}^{(10)}+\frac{1}{2}U_{m,ttt}^{(10)}(t_{01})^2+\bar{t}_0t_{01}U_{m,tt\tau}^{(10)}+\frac{1}{2}\bar{t}_0^{\,2}U_{m,t\tau\tau}^{(10)}+t_{01}U_{m,tt}^{(11)}+$$

$$\bar{t}_0U_{m,t\tau}^{(11)}+U_{m,t}^{(12)}+\varphi_m^{-\frac{1}{2}}t_{01}U_{m,t\tau}^{(10)}+\varphi_m^{-\frac{1}{2}}\bar{t}_0U_{m,\tau\tau}^{(10)}+\varphi_m^{-\frac{1}{2}}U_{m,\tau}^{(11)}=0 \tag{141}$$

$$(\varepsilon^3.1): U_{m,tt}^{(10)}t_{20}+\frac{1}{2}(t_{10})^2U_{m,ttt}^{(10)}+U_{m,t}^{(30)}-U_{3m,t}^{(30)}=0 \tag{142}$$

$$(\varepsilon^3.\delta): U_{m,tt}^{(10)}t_{21}+U_{m,t\tau}^{(10)}\bar{t}_{20}+\left(t_{10}t_{11}+t_{01}t_{20}\right)U_{m,ttt}^{(10)}+\bar{t}_0t_{20}U_{m,tt\tau}^{(10)}+t_{20}U_{m,tt}^{(11)}+\frac{1}{2}(t_{10})^2U_{m,ttt}^{(11)}+$$

$$t_{01}\left(U_{m,tt}^{(30)}-U_{3m,tt}^{(30)}\right)+\bar{t}_0\left(U_{m,t\tau}^{(30)}-U_{3m,t\tau}^{(30)}\right)+U_{m,t}^{(31)}-U_{3m,t}^{(31)}+\varphi_m^{-\frac{1}{2}}\mu_2't_{01}U_{m,tt}^{(10)}+ \tag{143}$$

$$\bar{t}_0\left(\mu_2'U_{m,t}^{(10)}\right)'_{,\tau}+\varphi_m^{-\frac{1}{2}}\mu_2'U_{m,t}^{(11)}+\varphi_m^{-\frac{1}{2}}t_{20}U_{m,t\tau}^{(10)}+\frac{1}{2}\varphi_m^{-\frac{1}{2}}(t_{10})^2U_{m,tt\tau}^{(10)}+U_{m,\tau}^{(30)}-U_{3m,\tau}^{(30)}=0$$

$$\left(\varepsilon^3.\delta^2\right): t_{22}U^{(10)}_{m,tt} + t_{21}U^{(10)}_{m,t\tau} + \left(\frac{1}{2}t_{11}^2 + t_{10}t_{12} + t_{02}t_{20} + t_{01}t_{21}\right)U^{(10)}_{m,ttt} +$$

$$\left(\bar{t}_{01}t_{20} + t_{11}\bar{t}_{20} + t_{01}\bar{t}_{20} + t_{21}\bar{t}_0 + \bar{t}_{10}t_{11} + \bar{t}_{11}t_{10}\right)U^{(10)}_{m,tt\tau} + t_{21}U^{(11)}_{m,tt} + \bar{t}_{20}U^{(11)}_{m,t\tau} +$$

$$t_{01}t_{20}U^{(11)}_{m,ttt} + \bar{t}_{10}t_{10}U^{(11)}_{m,tt\tau} + t_{20}U^{(12)}_{m,tt} + t_{02}\left(U^{(30)}_{m,tt} - U^{(30)}_{3m,tt}\right) + \bar{t}_{01}\left(U^{(30)}_{m,t\tau} - U^{(30)}_{3m,t\tau}\right) +$$

$$\frac{1}{2}t_{01}^2\left(U^{(30)}_{m,ttt} - U^{(30)}_{3m,ttt}\right) + \bar{t}_0 t_{01}\left(U^{(30)}_{m,tt\tau} - U^{(30)}_{3m,tt\tau}\right) + \frac{1}{2}\bar{t}_0\left(U^{(30)}_{m,t\tau\tau} - U^{(30)}_{3m,t\tau\tau}\right) +$$

$$\bar{t}_0\left(U^{(31)}_{m,t\tau} - U^{(31)}_{3m,t\tau}\right) + U^{(32)}_{m,t} - U^{(32)}_{3m,t} + \varphi_m^{-\frac{1}{2}}\mu_2' t_{02}U^{(10)}_{m,tt} + \varphi_m^{-\frac{1}{2}}\bar{t}_{01}\left(\mu_2' U^{(10)}_{m,t}\right)' +$$

$$\frac{1}{2}\varphi_m^{-\frac{1}{2}}\mu_2' t_{01}^2 U^{(10)}_{m,ttt} + \varphi_m^{-\frac{1}{2}}\bar{t}_0^2\left(\mu_2' U^{(10)}_{m,t}\right) + \varphi_m^{-\frac{1}{2}}\left\{\mu_2' t_{01}U^{(11)}_{m,tt} + \bar{t}_0\left(\mu_2' U^{(11)}_{m,t}\right)'\right\} +$$

$$\varphi_m^{-\frac{1}{2}}U^{(12)}_{m,t} + \varphi_m^{-\frac{1}{2}}\bar{t}_{20}U^{(10)}_{m,\tau\tau} + \varphi_m^{-\frac{1}{2}}t_{01}t_{20}U^{(10)}_{m,tt\tau} + \varphi_m^{-\frac{1}{2}}\left(\bar{t}_0 t_{20} + \bar{t}_{10}t_{10}\right)U^{(10)}_{m,t\tau\tau} +$$

$$\varphi_m^{-\frac{1}{2}}t_{20}U^{(11)}_{m,t\tau} + \frac{1}{2}\varphi_m^{-\frac{1}{2}}t_{10}^2 U^{(11)}_{m,tt\tau} + \varphi_m^{-\frac{1}{2}}t_{01}\left(U^{(30)}_{m,t\tau} - U^{(30)}_{3m,t\tau}\right) +$$

$$\varphi_m^{-\frac{1}{2}}\bar{t}_0\left(U^{(30)}_{m,\tau\tau} - U^{(30)}_{3m,\tau\tau}\right) + \varphi_m^{-\frac{1}{2}}\left(U^{(31)}_{m,\tau} - U^{(31)}_{3m,\tau}\right) = 0 \qquad (144)$$

Solving equations (139) to equation (144), the maximum displacement now becomes

$$U_a = \varepsilon\left[2\left(B_m\bar{a}_m\right) - \delta\pi\left(B_m\bar{a}_m\right)\right] + \varepsilon^3\left[\left\{3\varphi_m^{-1}\left(B_m\bar{a}_m\right) - \frac{1}{4}\varphi_m^{-1}\left(B_m\bar{a}_m\right)^3 V_3\right\} +\right.$$

$$\left.\delta\left\{-\frac{45}{32}\pi\varphi_m^{-\frac{1}{2}}\left(B_m\bar{a}_m\right)^3 + \pi\left(B_m\bar{a}_m\right)^3\varphi_m^{-1}\left(\frac{126}{32}\varphi_m^{-1} - \frac{315}{128}\right) + \frac{1}{4}\pi\varphi_m^{-1}\left(B_m\bar{a}_m\right)\left(V_4 + V_5\right)\right\}\right]$$

$$= \varepsilon\left[2\left(B_m\bar{a}_m\right) - \delta\pi\left(B_m\bar{a}_m\right)\right] + \varepsilon^3\left[\left\{3\left(B_m\bar{a}_m\right)\varphi_m^{-1}\left(1 - \frac{1}{2}V_3\right)\right\} -\right.$$

$$\left.\delta\pi\varphi_m^{-1}\left(B_m\bar{a}_m\right)^3\left\{\frac{45}{32}\varphi_m^{\frac{1}{2}} - \frac{126}{32}\varphi_m^{-1} + \frac{315}{128}\right\} + \frac{1}{4}V_6\right] + \ldots$$

$$= 2\left(B_m\bar{a}_m\right)\varepsilon\left[1 - \frac{\delta\pi}{2}\right] + 3\varepsilon^3\left(B_m\bar{a}_m\right)^3\varphi_m^{-1}\left[\left(1 - \frac{1}{12}V_3\right) -\right.$$

$$\left.\delta\pi\left\{\left(\frac{15}{32}\varphi_m^{\frac{1}{2}} - \frac{42}{32}\varphi_m^{-1} + \frac{105}{128}\right) + \frac{1}{12}V_6\right\}\right] + \ldots$$

(145)

Therefore,

$$U_a = 2\left(B_m\bar{a}_m\right)\varepsilon\left[1 - \frac{\delta\pi}{2}\right] + 3\varphi_m^{-1}\left(B_m\bar{a}_m\right)^3\varepsilon^3\left[1 - \delta\pi V_7 - V_8\right] + \ldots \qquad (146)$$

where

$$V_1 = \left\{\frac{5}{2\Omega_m^2} - \frac{15}{4\left(\Omega_m^2 - 1\right)} + \frac{3}{2\left(\Omega_m^2 - 4\right)} - \frac{1}{4\left(\Omega_m^2 - 9\right)}\right\}\cos\Omega_m t_0 \qquad (147)$$

$$V_2 = \frac{5}{2\Omega_m^2} + \frac{15}{4\left(\Omega_m^2 - 1\right)} + \frac{3}{2\left(\Omega_m^2 - 4\right)} + \frac{1}{4\left(\Omega_m^2 - 9\right)} \qquad (148)$$

$$V_3 = V_1 - V_2 \qquad (149)$$

$$V_4 = V_1 \qquad (150)$$

$$V_5 = -\frac{3}{\Omega_m^2} + \frac{21}{4\left(\Omega_m^2 - 1\right)} - \frac{3}{\left(\Omega_m^2 - 4\right)} + \frac{3}{4\left(\Omega_m^2 - 9\right)} \qquad (151)$$

$$V_6 = V_4 + V_5 \tag{152}$$

$$V_7 = \frac{15}{32}\varphi_m^{\frac{1}{2}} - \frac{42}{32}\varphi_m^{-1} + \frac{105}{128} \tag{153}$$

$$V_8 = -\frac{1}{12}\left(V_3 + \delta\pi V_6\right) \tag{154}$$

Again, for ease of further analysis,
let

$$U_a = \varepsilon q_1 + \varepsilon^2 q_2 + \varepsilon^3 q_3 + \ldots \tag{155}$$

where

$$q_1 = 2\left(B_m \bar{a}_m\right)\left[1 - \frac{\delta\pi}{2}\right] \tag{156}$$

$$q_2 = 0 \tag{157}$$

$$q_3 = 3\varphi_m^{-1}\left(B_m \bar{a}_m\right)^3\left[1 - \delta\pi V_7 - V_8\right] \tag{158}$$

As in Budiansky and Hutchinson (1964, 1966, 1972), the condition for dynamic buckling is

$$\frac{d\lambda}{dU_a} = 0 \tag{159}$$

As in Ette (2008, 2009), the series (155) can be expressed in the form

$$\varepsilon = \sum_{i=1}^{\infty} q_i U_a^i \tag{160}$$

Equating the coefficients of powers of ε , in (160),

$$e_1 = \frac{1}{q_1},\ \ e_3 = -\frac{q_3}{q_1^4} \tag{161}$$

where e_i depend on λ for $i = 1, 2, 3, \ldots$

The maximum displacement evaluated at $\lambda = \lambda_D$ is

$$U_a = \pm\sqrt{\frac{q_1^3}{3q_3}} \tag{162}$$

8. DYNAMIC BUCKLING LOAD λ_D

To determine the dynamic buckling load λ_D , express (160) as

$$\varepsilon = e_{1D} U_a + e_{3D} U_a^3 = \frac{2}{3}\sqrt{\frac{q_1}{3q_3}} \tag{163}$$

where

$$\lambda = \lambda_D, e_i = e_i\left(\lambda_D\right)$$

Further simplification of (163)

$$\varepsilon = \frac{2}{3}\left[\frac{2\varphi_m\left(B_m \bar{a}_m\right)\left(1 - \frac{\delta\pi}{2}\right)}{9\left(B_m \bar{a}_m\right)^3\left(1 - \delta\pi V_7 - V_8\right)}\right]^{\frac{1}{2}} \tag{164}$$

From equation (164),

$$\left(m^4 - 2m^2\lambda_D + 1\right)^{\frac{3}{2}} = \frac{9\lambda_D m^2 \bar{a}_m \varepsilon}{\sqrt{2}}\left[\frac{1 - \delta\pi V_9}{1 - \frac{\delta\pi}{2}}\right]^{\frac{1}{2}} \tag{165}$$

where

$$V_9 = \delta\pi\left(V_7 - \frac{1}{\delta\pi}V_8\right) \quad (166)$$

To find the dynamic buckling load λ_D at different modes m $(m = 1, 3, \ldots)$ and $\bar{a}_m = 0.01, 0.03, \ldots; m = 1, 3, \ldots$

Equation (164) for $m = 1, \bar{a}_1 = 0.01$, is

$$(1-\lambda_D)^{\frac{3}{2}} = 0.0225\lambda_D\varepsilon\left[\frac{1-\delta\pi V_9}{1-\frac{\delta\pi}{2}}\right]^{\frac{1}{2}} \quad (167)$$

9. Analysis

Equation (167) is evaluated numerically to obtain the various values of the dynamic buckling load for different values of the parameters, δ *and* ε .

Table 1: computed values of dynamic buckling load λ_D at various values of δ and various fixed values of imperfection $\varepsilon, m = 1, \bar{a}_1 = 0.01, \Omega^2_{m=1} = \dfrac{81-18\lambda+1}{2(1-\lambda)}$ for column

δ	λ_D at $\varepsilon = 0.01$	λ_D at $\varepsilon = 0.02$	λ_D at	λ_D at	λ_D at	λ_D at	λ_D at	λ_D at	λ_D at	λ_D at
0.00	0.991559	0.986639	0.982530	0.978879	0.975537	0.972423	0.969489	0.966701	0.964035	0.961474
0.01	0.994186	0.990659	0.987719	0.985104	0.982700	0.980474	0.978370	0.976369	0.974451	0.972610
0.02	0.994448	0.990863	0.987899	0.985269	0.982862	0.980613	0.978505	0.976495	0.974572	0.972725
0.03	0.994899	0.990899	0.988110	0.985456	0.982999	0.981000	0.978653	0.976633	0.974703	0.972847
0.04	0.995990	0.990999	0.988365	0.985673	0.983226	0.981000	0.978814	0.976783	0.974843	0.972979
0.05	0.995099	0.991999	0.988688	0.985927	0.983447	0.981151	0.978994	0.976949	0.974996	0.973121
0.06	0.995000	0.992999	0.988999	0.986243	0.983692	0.981375	0.979200	0.977132	0.975164	0.973264
0.07	0.995000	0.993099	0.989899	0.986649	0.983999	0.981633	0.979422	0.977335	0.975350	0.973446
0.08	0.995000	0.993000	0.989899	0.986999	0.984399	0.981942	0.979683	0.977559	0.975554	0.973633
0.09	0.995000	0.992099	0.989999	0.987999	0.984939	0.982322	0.979989	0.977826	0.975785	0.973840
0.10	0.995000	0.992000	0.989999	0.989900	0.985899	0.982799	0.979999	0.977999	0.975999	0.973999

Table 1: computed values of dynamic buckling load λ_D at various values of δ and various fixed values of imperfection $\varepsilon, m = 3, \bar{a}_3 = 0.03, \Omega^2_{m=3} = \dfrac{6562-162\lambda}{82-18\lambda}$ for column

δ	λ_D at $\varepsilon = 0.01$	λ_D at $\varepsilon = 0.02$	λ_D at	λ_D at	λ_D at	λ_D at	λ_D at	λ_D at	λ_D at	λ_D at
0.00	0.896784	0.842794	0.809000	0.765867	0.735499	0.708664	0.684507	0.662546	0.642410	0.623822
0.01	0.969450	0.952083	0.940000	0.925364	0.914098	0.910000	0.894073	0.884987	0.876406	0.868261
0.02	0.969369	0.951959	0.937760	0.925177	0.913882	0.910000	0.893809	0.884734	0.876103	0.867940
0.03	0.969288	0.951831	0.937517	0.924982	0.913661	0.910000	0.893542	0.884425	0.875719	0.867614

0.04	0.969205	0.951704	0.937183	0.924786	0.913436	0.910000	0.893267	0.884127	0.875480	0.867281
0.05	0.969120	0.951573	0.937014	0.924585	0.913200	0.910000	0.892990	0.883827	0.875160	0.866941
0.06	0.969034	0.951439	0.936836	0.924380	0.912974	0.910000	0.892705	0.883521	0.874853	0.866596
0.07	0.968947	0.951303	0.936836	0.924172	0.912735	0.910000	0.892409	0.883204	0.874520	0.866243
0.08	0.968857	0.951164	0.936656	0.923958	0.912492	0.910000	0.892121	0.882891	0.874187	0.865883
0.09	0.968770	0.951021	0.936474	0.923741	0.912244	0.910000	0.891816	0.882566	0.873819	0.865515
0.10	0.968672	0.950876	0.936287	0.923519	0.911990	0.910000	0.891511	0.882234	0.873481	0.865140

In order to investigate the numerical behavior of the dynamic buckling load of a geometrically imperfect column (viscously damped) under a time dependent load at different modes, the result presented in Table I and Table II would be very helpful in the analysis and summary of the results. Table I shows the variation of the damping value δ and dynamic buckling load λ_D with constant imperfection ε, and mode, $m=1$. Table II shows the variation of the damping and dynamic buckling load λ_D with constant imperfection ε, and mode, $m=3$ From the above tables, it show that increase in damping and decrease in the imperfection enhances the stability while the reverse destabilizes the system. As the mode increases, the dynamic buckling load decreases

10. Conclusion

Theoretical study of dynamic bucking load of a geometrically imperfect column lying on a nonlinear elastic foundation, trapped by a time dependent load has been carried out. Various values of the dynamic buckling load were obtained at different modes. From Table I and Table II, is clear that imperfection and damping have significant effects on the dynamic buckling load of a geometrically imperfect column at different modes. Finally, we observed that damping a system gives additional dynamic stability to any elastic structure in the dynamic buckling process. The mode $m=1$ is observed to be the dominant mode, while the mode $m=3$ is known as the reiferment mode. As the number of modes increases it reduces damping effect thereby decreasing the dynamic buckling load.

11. References

[1] Adhikari S. and Woodhouse J. (2000): Identification of damping: part 1, viscous damping, Journal of sound and vibrations, 243(1), 43-61.
[2] Ahmed Naif Al-Khazraji, Samir Ali A-Rabii and Hameed Shamkhi Al-Khazaali (2017): Improvement of dynamic buckling behavior of intermediate aluminized stainless steel column. Al-Khwarizmi Engineering Journal, Vol. 13, No. 1, p.p 26-41.
[3] Budiansky, B. and Hutchinson, J. (1964): Dynamic buckling of imperfection-sensitive structures. Pro. 12th internat.congr. Appl. Mech., Munich, 636-651.
[4] Budiansky, B. and Hutchinson, J. W. (1966): Dynamic buckling of elastic structures: criteria and estimates, in, Dynamic stability of structures, ed. G. Herrmann, Pergamon, New York.
[5] Budiansky, B. and Hutchinson, J. W. (1972): Buckling of circular cylindrical shell under axial compression. In: contribution to the theory of aircraft structures. Delft University Press, Netherlands, 239-260.
[6] Danielson, D. (1969): Dynamic buckling load of imperfection sensitive structures from perturbation procedures.
[7] Ette, A. M. (2005): On the dynamic buckling of a weakly damped nonlinear elastic model system under a slowly varying explicitly time dependent load, J. Nigerian Asso. Math. Physics, 9, 165-174.
[8] Ette, A. M. (2008): Perturbation technique on the dynamic stability of a lightly damped cylindrical shell axially stressed by an impulse. J. Asso. Math. Physics, 12, 103-120.
[9] Ette, A. M. (2009): Perturbation approach on the dynamic buckling of a lightly damped cylindrical shell modulated by a periodic load, J. Nigerian Math. Soc., 28, 97-135.
[10] Ette, A. M. and Osuji, W. I. (2007): Effects of static pre-loading on the dynamic Stability of a column on non-linear foundation stressed by a step load. J. Nigerian Asso. Math. Physics. 14(2), 323-332.
[11] Ette, A.M. and Udo-Akpam, I.V. (2016), Analysis of dynamic buckling of a model

structure with quadratic non linearity struck by a step load, superposed on quasi-static load. International Journal of Applied Science and Mathematics. Volume 3, issue 3, ISSN: 2394-2894

[12] Hutchinson, J. W. and Budiansky, B. (1966): dynamic buckling estimates. A.I.A.A J., 4(3), 525-530.

[13] Karagiozova, D. (2004): Dynamic plastic and dynamic progressive buckling of elastic-plastic circular shells-revisited. Latin American J. of solids and structures. 1, 423-441.

[14] Huyan, X. and Simitses, G. J. (1997): Dynamic buckling of imperfect cylindrical shells under axial compression and bending moment, A.I.A.A J. 35(8), 1404-1412.

[15] Lindberg, H. E., (2003): Little book of Dynamic Buckling, WWW.lindberglce.com/tech/bulkbook.htm.

[16] Nima Aghdam and Kai-Uwe Scroeder (2021): Dynamic buckling of crash boxes under an impact load. Proceedings of the 8th International conference on coupled instabilities in MetalStructures. (CIMS 2021)

[17] Onuoha, N.O. and Ette A.E. (2017): Asymptotic analysis of the dynamic stability of a viscously damped elastic model structure under step load. Journal of the Nigerian Association of Mathematical Physics. Volume 41, pp 131 – 140.

[18] Sapsis, T. P., Quinn, D. D., Vakakis, A. F and Bergman, L. A. (2012): Effective stiffening and damping enhancement of structures with strongly nonlinear local attachments, J. of Vibrations and Acoustics, 134, 1-12.

[19] Simitses, G. J. (1983): Effects of preloading on the dynamic stability of structures, A.I.A.A J., 21 1174-1180.

[20] Simitses, G. J. (1989): Dynamic Stability of suddenly loaded structures, Springer-Verlag, New York.

[21] Song-Hak U, Yong-II So, and Wang-Myong So (2022): International Journal of Structural Stability and Dynamics,. Vol. 22, No. 08, 2250086

[22] Svalbonas, V. and Kalnins, A. (1977): dynamic buckling of shells; evaluation of various methods, T. M. Nuclear Engineering and Design, 44(3), 331-356.