Exploring Learner's Misconceptions in Algebraic Problem Solving: Insights for Effective Instruction and Assessment

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Abstract
This qualitative research focused on Seventh grade students' misconceptions during algebraic problem solving. 15 open-ended algebraic tasks were developed, standardised and administered to 12 Seventh grade school students in Lagos State. Test items were developed with focus on the highlighted possible algebraic misconceptions as identified in the literature review. After the test administration, each item and student's solutions were categorised by coding the misconceptions under the identified possible misconceptions in the literature review. Students' solutions were observed, organised, coded, categorised and discussed. Findings revealed that, mostly observed misconceptions among the solutions for the algebraic tasks were (a) interpreting the variable using a specific unknown number or generalised number; (b) perceiving the equal sign more as an indication of the outcome of an arithmetic operation rather than recognizing it as a representation of mathematical equivalence; (c) misconception in operation of simplification, and simplifying algebraic expressions.

Keywords: Algebra, misconception, equations.

INTRODUCTION

Increasing teachers' awareness of common students misconceptions in Mathematics is likely to result in a transformative impact on their instructional methods therefore, understanding learners' errors and misconceptions of algebraic concepts during instruction is important. More specifically, their errors and misconceptions with algebraic variables and labels. One major tool a teacher can employ in gaining insight of a learner's level of understanding of algebraic concepts is through the use of an assessment. In this review, I will explore the various misconceptions during algebraic problem solving towards providing insight for effective instruction..

In my years of teaching, I have witness learners' algebraic misconceptions in varying ways in the classroom which often include; difficulty in the understanding of variables; interpreting variables, literal symbols, and unknown quantities; construing a variable as a label for an object; interpreting algebraic expressions, and comprehending an algebraic equations and algebraic expressions; interpreting the wordings of the algebraic items, and the general relationships that exist between these algebraic terms and concepts; and many more. Even though student's understanding of theses concepts (especially the ability to use variables to represent unknowns or varying quantities; or the understanding of the equal sign to represent a relation between two equivalent quantities) are of immense benefit starting from the early grade level (Knuth et al 2016), various literatures also suggest that students still have misconceptions in areas like operational symbols, simplifying expressions, and solving equations.

Algebra is a foundational component of mathematics education, therefore when learners are able to understand and manipulate algebraic concepts, it provides them with the fundamental building block of algebraic thinking, which supports the grasp of higher-level mathematical concepts. Guiding learners to reach this level of algebraic understanding is often challenging since learners are often posed with different misconceptions during algebraic instruction.

REVIEW OF RELATED LITERATURE

Research has revealed various misconceptions students develop that hinder their algebraic reasoning and problem solving abilities. For example, Khalid, Yakop, F., & Ibrahim, H., (2020) investigates how students interpret letters and symbols when solving algebraic problems. Through a test, observations, and interviews, the researchers identified themes of students' misconceptions to include: "combining" letters during addition, equating letters as 1, misinterpreting the equal sign, inconsistently applying their own incorrect rules, and seeing letters as abbreviations for objects. This review synthesises research findings on common student misconceptions in algebraic problem solving toward providing insights for effective teaching and learning, as well as assessment.

Misconceptions About Variables and Notations

One challenge for students learning algebra is grasping the notion of variables as generalised numbers rather than unknowns or objects (Küchemann, 1981; Stacey & MacGregor, 1997; Stephens, 2005). In this category of student's interpretation, Kuchemann (1981) noted that (1) the letter is ignored, given an arbitrary value, or used as the name of an object (2) the letter is used as a specific unknown number or generalised number. In support of Kuchemann's findings of students ignoring the letter, Moss et al 2018 presented us with Anna's solution as she wrote the expression 12p + 20h = 32 for the total number of polygons when presented with a miniature soccer balls and asked to count the number of hexagons and the number of pentagons and then write an expression for the total number of polygons. This indicates that Anna approached the task by treating the variables "p" and "h" as having no assigned values, and interpreted the equal sign as a cue to calculate the sum of the coefficients.

Stacey & MacGregor, 1997) observed that students often rely on intuition, guessing, analogies with familiar symbol systems, or misinformation from misleading teaching materials when interpreting letters and algebraic expressions.

The persistent misconception of misinterpreting algebraic letters as abbreviated words or labels for objects has been documented in various research findings. It is possible that when students are familiar that concepts in applied mathematics are usually denoted by the initial letters of their names (e.g., a implies area, b implies breadth, l implies length, m implies mass, t implies time, etc.), It is likely that this usage of letters likely reinforces the belief that letters in mathematical expressions and formulas represent words or objects rather than numerical values . Stacey & MacGregor, 1997) in their studies, reported (1) how student perceive that letter is a label associated with the name of an object (e.g., C to mean 'Con's height' and D to mean 'David's height' in C + 10 = D), (2) letter equals 1 unless otherwise specified (e.g., 10 + h = 11), and (3) letter has a general referent that includes various specifics (h means 'height', so it means both 'David's height' and 'Con's height' in the statement h = h + 10).

Another illustration of Students constructing variables simply as labels or as representing a single unknown value rather than seeing them as quantities that can represent different values in varied contexts has also been reported by (Rosnick, 1981; Clement, 1982). In their studies, a population of 150 entering engineering students at the University of Massachusetts were presented with a mathematical task to write an equation, using the variables *S* and *P* to represent the following statement:

"At this university there are six times as many students as professors."

(Use S for the number of students and P for the number of professors)

Fully 37% were unable to write the correct equation. The studies documented the most common error to be *6S = P* instead of *S = 6P*.

Steinle et al., (2009) classifies Küchemann (1981); MacGregor & Stacey (1997) evidence as *non-numerical* interpretations of letters, for example, letter ignored and letter as object and further contrasted the former with incorrect *numerical* ways of thinking where the letter is assigned a numerical value, but having incorrect ideas about what values the numbers can take. Such *numerical* misconceptions can be seen when students associate a number according to the position of a letter in the alphabet (a = 1, b = 2, c = 3, or x = 24 and y = 25 etc), and those who believe that different letters must represent a different number. This limited view of variables impacts students' ability to interpret algebraic expressions and work flexibly with equations.

Misconceptions About Equivalence

A greater number of students exhibit a limited grasp of the meaning of the equal sign. Often, they perceive the equal sign more as an indication of the outcome of an arithmetic operation instead of recognizing it as a representation of mathematical equivalence (Baroody & Ginsburg, 1983; Behr et al., 1980; Kieran, 1981; Rittle-Johnson & Alibali, 1999). Related to the idea of variables, students often fail to see algebraic equations as statements of relational equivalence between two expressions, rather than processes to obtain a solution.

Knuth et al., (2006) noted that many students exhibit a limited comprehension of the equal sign, and this understanding correlates with their performance on tasks involving solving algebraic equations. In their studies, they examine the relation between grade level (6, 7, or 8) and the likelihood of exhibiting a relational understanding of the equal sign. Results reveal that students were not more likely to exhibit a relational understanding of the equal sign as grade level increased, nor was there a U-shaped pattern in students' likelihood of exhibiting a relational understanding across the grade levels. Alibali et al., (2007) findings from a longitudinal study that examined middle school students' understanding of the equal sign, the relationship between their understanding and their performance on problems using the equal sign also provided evidence that many students do not exhibit a relational understanding of the equal sign.

Misconceptions About Unknowns

When students first learn about unknowns, they often think of them as having a specific meaning instead of understanding that they are just symbols representing any number in general. They interpret unknowns as standing for particular objects rather than as placeholders for values to be determined. This impedes flexibility in working with unknowns in equations. Linchevski & Herscovics (1996) used equations with only one occurrence of the unknown (e.g. $ax + b = c$) and equations with two occurrences of the unknown on the same sides (e.g. $ax + bx = c$) and on different sides of the equal sign (e.g. $ax + b = cx + d$) to examine the shift in students' procedures. It was found that students could spontaneously group terms that were purely numeric rather than terms in the unknown, which mean "students could not operate spontaneously with or on the unknown".

Misconceptions About Expressions

Studies (MacGregor & Stacey,1997) show that students have difficulties interpreting algebraic expressions, distinguishing them from equations, and understanding the meaning of operations in expressions . They tend to interpret expressions in rigid operational ways rather than viewing them relationally, which impacts their ability to meaningfully compose and decompose expressions.

Booth (1988) noted that when learning arithmetic, students are often taught to write answers in a single term, and leaving the answer for example as, "3 + 5" was not considered a correct answer. As a result, students often misconceive that "x + y" could equal the total number of items in two sets. This lack of understanding leads to difficulty for students to see "a + b" as a mathematical concept in algebra.

Greeno (1982) conducted a study with beginning algebra students to test their conception of structure of relations in problems. Findings reveal that students were short of structural understanding of algebra. For example, students would split up algebraic expressions into distinct, disconnected pieces without recognizing how terms interacted within the expression. Their simplification process appeared to be somewhat arbitrary. For instance, the expression $3(2a-7b) + 2a$ is treated as $3(2a - 7b + 2a)$ at one point and as $3(2a+2b) - 7b$ at another point.

Wenger (1987) highlighted the students' use of arbitrary strategies in simplification. This was attributed to their difficulty in recognizing the appropriate elements within algebraic expressions. Furthermore, students struggled to apply their knowledge of simplification from one context, to another context.

Concatenation emerged as another common mistake in the process of simplifying algebraic expressions. For example, $25a - 7$ is seen to be concatenated by students into $18a$ and $3ab - 3a$ concatenated into b .Carry et al. (1980) affirmed that this type of error was not exclusive to beginning algebra students but was also observed among college students. Their study identified this error as the most prevalent one made by students when simplifying expressions at various stages of equation solving. Moreover, they suggested that this error might result from students over-generalizing specific validated operations to achieve a more generic operation. Consequently, the arbitrary strategies and concatenation approaches represent typical behaviors exhibited by students during the simplification of expressions.

Christou et al., 2007, 2022; Christou & Vosniadou, 2009, 2012; McNeil, et al., (2010); Van Dooren et al., (2010) noted that students often misinterpret literal symbols to stand for natural numbers rather than for any (real) number (integrity effect). Students also tend to assign to the values of a given algebraic expression an invariant sign (phenomenal sign effect). As noted by Christou et al., (2022), algebraic expressions contain literal symbols that represent numbers, which is precisely the reason why one could expect natural numbers bias (NNB) effects on students' interpretations of algebraic expressions. In their studies consisting of 138 students (8th and 9th grades) from two Greek middle-class urban public high schools, results revealed that there is indeed a dual NNB effect on students' interpretations of algebraic expression. Students also performed significantly higher when the algebraic expression had a positive phenomenal sign (e.g., k + 3), than when the algebraic expression had a negative phenomenal sign (e.g., -d-4). The effects of natural number assumptions on students' interpretations of algebraic expressions can not be overemphasised, as it constrains students' interpretation of algebraic expressions. Therefore, there is a need to recognise this manner of students' interpretations of algebraic expressions.

Another challenging part of algebra especially in the lower level may be the confusion regarding the combination of letters and numerals in the same mathematical expression as reported in Brizuela, Blanton, M., Sawrey, K., Newman-Owens, A., & Murphy Gardiner, A., (2015). Such misconception can be seen when a student erroneously conceptualises an expression such as 2x + 5 as 2 + 5, ignoring the "x - variables". Sometimes, the variable may be conceptualised as an abbreviated word, example; 2b + 3 may be viewed as 2boys + 3. In another way, students may perceive an expression such as 2x + 3y to mean 5xy.

Misconceptions About Equations

Moss & Lamberg, T., (2019) highlights the development of algebraic thinking in their studies by investigating the learning trajectory of 6th grade students in developing understanding of algebraic expressions and equations. The researchers identified five levels of thinking that emerged based on different problem types: Label Thinker, Formulaic Thinker, Substituter, Solver, and Functional Thinker. These levels reflected students' evolving interpretation of letters as labels, known values, unknowns, and variables. This supports the fact that student understanding of how letters and variables are used in different contexts is critical for their conceptual development.

A key struggle for students in comprehending algebraic equations is letting go of familiar arithmetic practices (Kieran, 1992; Herscovics & Linchevski, 1994). Students interpret the operations in equations procedurally to find "the answer" rather than grasping the relational nature of equations (Carpenter et al., 2005). This reliance on arithmetic thinking hinders students' ability to flexibly solve equations using methods like undoing through addition/subtraction of equivalent terms.

Research indicates that the equal sign is frequently misinterpreted by students across all levels of education. However, high school and college students may be more inclined to view the equal sign as a formal symbol for equivalence compared to younger students (Welder, 2012). Behr et al., (1980) revealed that beginning algebra students took the equal sign as a procedural indicator. For example, students were reluctant to accept expressions such as 7 + 3 = 6 + 4 or 6 = 6. They would like to change equality 7 + 3

= 6 + 4 to be separated into two equalities 7 + 3 = 10 and 6 + 4 = 10; equality 3 + 0 = 3 (Welder, 2012) in terms that they would think that the right side should be the answer. In addition, Falkner et al., (1999) further offered specific data for such limited interpretation of the equal sign. In their investigation, all the participants (145 American students from grade 6) could not correctly fill the number sentence 8 + 4 = __ + 5. The typical answer for this question was 12 or 17.

Li, Ding, Capraro and Capraro (2008) in a similar finding reveals that there were only 25 out of 105 American Grade 6 students who could correctly fill the first blank in such a number sentence 3 + __ = 4 + 4 = ___ . However, 91 out of 105 students could give a correct answer 8 for the second blank. In a word, the misunderstanding and ill operation of equal sign impede students from access to the concept of equity which is the core component of the concept of equation in algebra learning.

Building on previous research, the primary goal of this study was to explore the misconceptions that students encounter while solving algebraic equations. Through this investigation, the aim was to contribute positively to algebra lessons by shedding light on the 8th-grade students' misconceptions in their approaches to solving algebraic problems.

RESEARCH QUESTION

What misconceptions do Seventh-grade students participating in the study have in their approach to solving algebraic problems?

METHODOLOGY

To address the research question, a qualitative approach was employed in this study, involving 12 Seventh-grade students. Convenience sampling was utilised as the sampling method, chosen for its accessibility and the ease of obtaining official permission from the school administration. The participants were drawn from a middle school with 30 Seventh-grade students. Among the participants, 5 were boys, and 7 were girls. Also, 3 students were classified as low-achievers in mathematics, 3 as middle-achievers, and 6 as high-achievers based on the students' academic performance in mathematics lessons during the current academic year.

INSTRUMENT

In order to identify the misconceptions of the participant students, 15 open-ended algebra questions were developed. The mathematics curriculum and the syllabus offered by the Ministry of Education were used as a rationale for developing suitable items for the 7th grade students. Expert opinion was obtained from mathematics teachers and test experts. The instrument was trial tested and the final items were adjusted appropriately and standardised with a reliability coefficient of .82.

PROCEDURE

The research investigated the misconceptions of the Seventh grade students in their algebra problem solving. Test instruments with 15 items were administered to 10 eighth grade students. The duration of the test was flexible to allow students to explore each task. The questions were answered in approximately 1 hour. The general procedures of test development, validation, administration and scoring as identified in scientific literature were followed.

After the data collection, categorization was made by coding the misconceptions of the students. Answers were organised, coded, categorised and discussed. For the data analysis, misconceptions synthesised from literature review were looked out for in the students' solutions. Such misconceptions included but not limited to; Ignoring the presence of the letters; comprehending the letters as objects; Thinking that letters always have one specific value; Thinking that the letters can only stand for the natural numbers; Taking no notice for the negative/positive signs while manipulating the algebraic expressions etc. Student's answers, approaches, and workings were coded in the suitable places based on the synthesised misconceptions from the literature review. The frequency distributions of the coded results were constructed in a tabular form.

RESULTS

In Table 1, the misconceptions synthesised from literature review, the questions in which the students' solutions showed misconception and the number of the participants who did the misconception are represented.

Table 1. Misconceptions and the distribution of the misconceptions

Algebraic misconceptions	**Question number**	**No. of the participants**
Misconception grasping the notion of variables as generalised numbers rather than unknowns or objects	Q1, Q3, Q8	4
Ignoring the letter, assigning an arbitrary value, or used as the name of an object	Q1, Q2, Q7, Q8, Q10	3
the letter is used as a specific unknown number or generalised number	Q3, Q8, Q11	5
Interpreting letters and algebraic expressions on intuition, guesses, and comparisons to familiar symbolic systems when attempting to make sense of letters and algebraic expressions.	Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q10	2
Misinterpretation of algebraic letters as abbreviations for words or as labels	Q1, Q2, Q3, Q4, Q6, Q10	4
associating a number according to the position of letter in the alphabet (a = 1, b = 2, x = 24 and y = 25 etc),	Q1, Q2, Q3, Q4, Q5, Q7, Q8, Q10, Q11,	0

perceiving the equal sign more as an indication of the outcome of an arithmetic operation rather than recognizing it as a representation of mathematical equivalence	Q1, Q3, Q4, Q5, Q6, Q7, Q11, Q12	5
misinterpreting algebraic expressions, distinguishing them from equations,	Q1, Q2, Q3, Q5, Q8, Q10, Q11, Q12	0
Misconception in operation of simplification, and simplifying algebraic expressions	Q5, Q6, Q7, Q8, Q9, Q12	5
perceiving an expression such as $4x + 2y$ to mean 6xy.	Q8, Q10, Q7, Q8	4
Misconception of ignoring or disregarding positive and negative signs or operational signs as they carried out procedural steps	Q7, Q8, Q9, Q11, Q12	3

From Table 1, No student associates a number according to the position of letters in the alphabet (a = 1, b = 2, x = 24 and y = 25 etc) or misinterpret algebraic expressions, distinguishing them from equations. Four (4) students in each group show (a) a misconception in grasping the notion of variables as generalised numbers rather than unknowns or objects; (b) a misinterpretation of algebraic letters as abbreviated words or labels; (c) perceiving an expression such as $4x + 2y$ to mean 6xy. Three (3) students in each group show misconceptions (a) where the letter is ignored, given an arbitrary value, or used as the name of an object; (b) taking no notice for the negative/ positive signs while manipulating the algebraic expressions. Five (5) students in each group (a) interpret the letter using a specific unknown number or generalised number; (b) perceive the equal sign more as an indication of the outcome of an arithmetic operation rather than recognizing it as a representation of mathematical equivalence; (c) show misconception in operation of simplification, and simplifying algebraic expressions.

DISCUSSION

The results shows that the students do not necessarily associates a number according to the position of letters in the alphabet (a = 1, b = 2, x = 24 and y = 25 etc) or have challenge misinterpret algebraic expressions, or distinguishing them from algebraic equations. Though other misconceptions are not overlooked, the mostly observed misconceptions among the participants' solutions for the algebraic tasks were (a) interpreting the letter using a specific unknown number or generalised number; (b) perceiving the equal sign more as an indication of the outcome of an arithmetic operation rather than recognizing it as a representation of mathematical equivalence; (c) misconception in operation of simplification, and simplifying algebraic expressions.

Interpreting the letter using a specific unknown number or generalised number is also presented in Küchemann, 1981; Rosnick, 1981; Stacey & MacGregor, 1997; Stephens, 2005. Perceiving the equal sign more as an indication of the outcome of an arithmetic operation rather than recognizing it as a representation of mathematical equivalence is also reported in Alibali et al., (2007); Baroody & Ginsburg, (1983); Behr et al., (1980); Kieran, (1981); Knuth et al., (2006); Rittle-Johnson & Alibali, (1999) where students fail to see algebraic equations as statements of relational equivalence between two expressions, rather than processes to obtain a solution. Lastly, misconception in the operation of simplification, and simplifying algebraic expressions is related to the findings reported in Carry, Lewis, & Bernard (1980); Greeno (1982); Wenger (1987). The students employed seemingly random approaches when attempting to simplify expressions. A potential explanation for this behaviour is that the students struggled to identify productive steps to take or lacked the ability to properly apply previously learned rules for simplification.

CONCLUSION AND IMPLICATIONS

This review highlights the diverse misconceptions that students often have during algebraic problem solving, and classify them under the following groups: 1. Misconceptions about Variables 2. Misconceptions about equivalence 3. Misconceptions about Unknowns 4. Misconceptions about Expressions, and 5. Misconceptions about Equations. Appreciating these misconceptions can help inform instructional design and pedagogical approaches to build students' algebraic reasoning in a more meaningful way. Further research on effective strategies to identify and remediate these misconceptions promises to improve outcomes in students' algebraic learning.

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